Heat Production of Noninteracting Fermions Subjected to Electric Fields

JEAN-BERNARD BRU
Universidad del País Vasco and Ikerbasque, Basque Foundation for Science

WALTER DE SIQUEIRA PEDRA
Universidade de São Paulo

AND

CAROLIN HERTLING
Johannes Gutenberg-Universität Mainz

Abstract

Electric resistance in conducting media is related to *heat* (or *entropy*) production in the presence of electric fields. In this paper, by using Araki's relative entropy for states, we mathematically define and analyze the heat production of free fermions within external potentials. More precisely, we investigate the heat production of the nonautonomous C^*-dynamical system obtained from the fermionic second quantization of a discrete Schrödinger operator with bounded static potential in the presence of an electric field that is time- and space-dependent. It is a first preliminary step towards a mathematical description of transport properties of fermions from thermal considerations. This program will be carried out in several papers. The regime of small and slowly varying in space electric fields is important in this context and is studied the present paper. We use tree-decay bounds of the n-point, $n \in 2\mathbb{N}$, correlations of the many-fermion system to analyze this regime. We verify below the first law of thermodynamics for the system under consideration. The latter implies, for systems doing no work, that the heat produced by the electromagnetic field is exactly the increase of the internal energy resulting from the modification of the (infinite volume) state of the fermion system. The identification of heat production with an energy increment is, among otheg things, technically convenient. We initially focus our study on noninteracting (or free) fermions, but our approach will be later applied to weakly interacting fermions. © 2015 Wiley Periodicals, Inc.

Contents

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1 Introduction

Ohm and Joule's laws, respectively derived in 1827 and 1840, are among the most resilient laws of (classical) electricity theory. In standard textbooks, the microscopic theory presented to explain Ohm's law is based on the Drude model proposed in 1900, before the emergence of quantum mechanics. In this model, the motion of electrons and ions is treated classically and the interaction between these two species is modeled by perfectly elastic random collisions. This quite elementary model explains very well DC- and AC-conductivities in metals, qualitatively. There are also improvements of the Drude model taking into account quantum corrections. Nevertheless, to our knowledge, there is no rigorous microscopic (complete) description of the phenomenon of linear conductivity from first principles of quantum mechanics. It is a highly nontrivial question. Indeed, problems are in this case doubled because the electric resistance of conductors results from both the presence of disorder in the host material and interactions between charge carriers.

Rigorous quantum many-body theory is a notoriously difficult subject. The hurdles that have to be overcome in order to arrive at important new mathematical results involve many different fields of mathematics such as probability theory, operator algebras, differential equations, and functional analysis. Disorder leads us to consider random Schrödinger operators like the celebrated Anderson model. It is an advanced and relatively mature branch of mathematics. For instance, it is known that, in general, the one-dimensional Anderson model only has a purely point spectrum with a complete set of localized eigenstates (Anderson localization), and it is thus believed that no steady current can exist in this case. For more details, see, e.g., [19]. Nevertheless, even in the absence of interactions, there are, to our knowledge, only a few mathematical results on transport properties of such models that yield Ohm's law in some form.

Indeed, Klein, Lenoble, and Müller introduced for the first time in [20] the concept of a "conductivity measure" for a system of noninteracting fermions subjected to a random potential. More precisely, the authors considered the Anderson tight-binding model in the presence of a time-dependent spatially homogeneous electric field that is adiabatically switched on. See also [5] for further details on linear response theory of such a model. The fermionic nature of charge carriers—electrons or holes in crystals—was implemented by choosing the Fermi-Dirac distribution as the initial density matrix of particles.[1] In [20] only systems at zero temperature with Fermi energy lying in the localization regime are considered, but it is

[1] This corresponds to $t \to -\infty$ in their approach.

shown in [21] that a conductivity measure can also be defined without the localization assumption and at any positive temperature. Their study can thus be seen as a mathematical derivation of Ohm's law for space-homogeneous electric fields having a specific time behavior. [8] is another mathematical result on free fermions proving Ohm's law for graphene-like materials subjected to space-homogeneous and time-periodic electric fields. Observe, however, that Joule's law and heat production are not considered in [8, 20, 21].

We propose in a companion paper a different approach to the conductivity measure based on a natural thermodynamic principle, the positivity of the heat (or entropy) production, together with the Bochner-Schwartz theorem [25, theorem IX.10]. Our aim is to derive both Ohm's and Joule's laws for the Fourier components of time-dependent electric fields from the analysis of the heat production in a realistic many-fermion system. We first focus our study on *noninteracting* (or free) fermions in the presence of disorder, here a static external potential, while keeping in mind its possible extension to interacting fermions. Indeed, the possibility of naturally extending results to systems with interaction is one of the main advantages and motivations of the approach we propose here. This will be discussed in more detail in subsequent papers. Therefore, although there is no interaction between fermions, we do *not* restrict our analyses to the one-particle Hilbert space to study transport properties. Instead, our approach is based on the algebraic formulation of many-fermion systems on lattices.

As observed by J. P. Joule [18] in the original paper, the electric resistance is associated with heat production in the conducting system. Therefore, the first step is to rigorously define and analyze the concept of *heat* production induced by electric fields on the fermion system. This study is the main subject of the present paper. At constant temperature, the heat production is, by definition, a quantity that is proportional to the *entropy* production. The proportionality coefficient is the temperature of the system. In order to give a precise mathematical definition of this quantity, we use in Section 3.1 Araki's relative entropy [2, 3, 22], which, in our case, turns out to be finite for all times. The latter uses the concept of spatial derivative operators [10]; see Section A.1.

Part of the paper is devoted to recovering the first law of thermodynamics for the system under consideration, implying that the heat production generated by the electromagnetic field is exactly the increase of the *internal* energy resulting from the modification of the (infinite volume) state of the system. An increment of internal energy of the system is defined here as being the increase of total energy minus the increase of potential energy associated with the external electric field. See Sections 3.2. The first law of thermodynamics is an important outcome in our context because it leads to more explicit expressions for the heat production. Moreover, the increase of *total* energy (i.e., internal plus potential energy) of the *infinite* system obeys a principle of conservation and is exactly the work performed by the electric field on the charged particles; see Section 3.2. This is well-known for dynamics on C^*-algebras generated by time-dependent bounded symmetric derivations; see, for instance, discussions in [7, section 5.4.4.]. Here we prove a version of that result for our particular *unbounded* case.

Note that Ohm's law corresponds to a linear response to electric fields. We thus rescale the strength of the electromagnetic potential by a real parameter $\eta \in \mathbb{R}$ and will eventually take the limit $\eta \to 0$ (in a subsequent paper). Understanding the behavior of the heat production as a function of η is a necessary step in order to obtain Ohm's and Joule's laws. By using the fact mentioned above that the heat production can be expressed in terms of an energy increment (Section 3.2), it can be shown that the heat production is a real analytic function of the scaling parameter η. The coefficients of the (absolutely convergent) power series in η for the heat production have the following important property: They behave, at any order $k \in \mathbb{N}$, like the volume of the support (in space) of the applied electric field, as physically expected. Such a behavior permits us, in particular, to define densities (like heat production per unit volume). We remark that naive bounds only predict that the k-coefficient of the power series for the heat production should behave like the k-power of the volume of the support of the applied electric field. However, the heat production is proven to behave like η^2 times the volume of the support of the applied electric field provided $|\eta|$ is sufficiently small. This is done in Section 5.5; see also Section 3.3. Moreover, this result makes possible the study of nonquadratic and nonlinear corrections to Joule's law and Ohm's law, respectively.

To obtain the properties described above for the power series in η representing the heat production, we use a pivotal ingredient, namely *tree-decay bounds* on multicommutators. These bounds are derived in Section 4 and are useful to analyze multicommutators of monomials in annihilation and creation operators. They will also be necessary in subsequent papers.

To conclude, our main assertions are Theorems 3.2 and 3.4 and Corollary 4.3. This paper is organized as follows:

- In Section 2 we describe nonautonomous C^*-dynamical systems for (free) fermions associated to discrete Schrödinger operators with bounded (static) potentials in the presence of an electric field that is time- and space-dependent.
- Section 3 introduces the concept of heat production and discusses its main properties.
- Section 4 is devoted to tree-decay bounds for expectation values of multicommutators.
- All technical proofs related to Section 3 are postponed to Section 5.
- The Appendix containing two parts: Section A.1 is a concise overview of quantum relative entropy [2, 3, 22]. In Section A.2 it is shown that all properties of the infinite system we use here result from the corresponding ones of finite systems, at large volume. Note that Section A.2 is not really used in other sections and is supplementary only.

Note (Generic Constants). To simplify notation, we denote by D any generic positive and finite constant. These constants do not need to be the same from one statement to another.

2 C^*-Dynamical Systems for Free Fermions

2.1 CAR C^*-Algebra

The host material for conducting fermions is assumed to be a cubic crystal. Other crystal families could also be studied in the same way, but, for simplicity, we refrain from considering them. The unit of length is chosen such that the lattice spacing is exactly 1. We thus use the d-dimensional cubic lattice $\mathfrak{L} := \mathbb{Z}^d$ $(d \in \mathbb{N})$ to represent the crystal and we define $\mathcal{P}_f(\mathfrak{L}) \subset 2^{\mathfrak{L}}$ to be the set of all *finite* subsets of $\mathfrak{L}$.

Within this framework, we consider an *infinite* system of charged fermions. To simplify notation we only consider spinless fermions with *negative* charge. The cases of particles with spin and/or positively charged particles can be treated by exactly the same methods. We denote by $\mathcal{U}$ the CAR algebra of the infinite system. More precisely, the (separable) C^*-algebra $\mathcal{U}$ is the inductive limit of the finite-dimensional C^*-algebras $\{\mathcal{U}_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathfrak{L})}$ with identity $\mathbf{1}$ and generators $\{a_x\}_{x \in \Lambda}$ satisfying the canonical anticommutation relations: For any $x, y \in \mathfrak{L}$,

$$a_x a_y + a_y a_x = 0, \quad a_x a_y^* + a_y^* a_x = \delta_{x,y} \mathbf{1}. \tag{2.1}$$

2.2 Dynamics in the Presence of Static External Potentials

It is widely accepted that electric resistance of conductors results from both the presence of disorder in the host material and interactions between charge carriers. Here we only consider effects of disorder for noninteracting fermions. That means physically that the particles obey the Pauli exclusion principle but do not interact with each other via some mutual force. This setup corresponds, for example, to the case of low electron densities. Our approach can be applied to weakly interacting fermions on the lattice, but the analysis would be—from the technical point of view—much more demanding of course.

Disorder in the crystal will be modeled in subsequent papers by a random external potential coming from a probability space $(\Omega, \mathfrak{A}_\Omega, \mathfrak{a}_\Omega)$ with $\Omega := [-1,1]^{\mathfrak{L}}$. In the present work, however, all studies are performed at any fixed $\omega \in \Omega$, and all the results will be uniform with respect to the choice of $\omega \in \Omega$. Note that, for any $\omega \in \Omega$, $V_\omega \in \mathcal{B}(\ell^2(\mathfrak{L}))$ is by definition the self-adjoint multiplication operator with the function $\omega : \mathfrak{L} \to [-1,1]$. The static external potential V_ω is of order $\mathcal{O}(1)$, and we rescale its strength by an additional parameter $\lambda \in \mathbb{R}_0^+$ (i.e., $\lambda \geq 0$); see (2.4).

For any function $\omega \in \Omega$, we define the dynamics of the lattice fermion system via a strongly continuous (quasi-free) group of automorphisms of the C^*-algebra $\mathcal{U}$. To set up this time evolution, we first define annihilation and creation operators of (spinless) fermions with wave functions $\psi \in \ell^2(\mathfrak{L})$ by

$$a(\psi) := \sum_{x\in\mathfrak{L}} \overline{\psi(x)} a_x \in \mathcal{U}, \quad a^*(\psi) := \sum_{x\in\mathfrak{L}} \psi(x) a_x^* \in \mathcal{U}. \tag{2.2}$$

These operators are well-defined because of (2.1). Indeed,

$$\|a(\psi)\|^2, \|a^*(\psi)\|^2 = \|\psi\|_2^2, \quad \psi \in \ell^2(\mathfrak{L}), \tag{2.3}$$

and thus the antilinear and linear maps $\psi \mapsto a(\psi)$ and $\psi \mapsto a^*(\psi)$), respectively, from $\ell^2(\mathfrak{L})$ to $\mathcal{U}$ is norm-continuous. Clearly, $a^*(\psi) = a(\psi)^*$ for all $\psi \in \ell^2(\mathfrak{L})$.

Now, for any function $\omega \in \Omega$ and strength $\lambda \in \mathbb{R}_0^+$ of the static (external) potential, we define the free dynamics via the unitary group $\{\mathrm{U}_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$, where

$$\mathrm{U}_t^{(\omega,\lambda)} := \exp(-it(\Delta_{\mathrm{d}} + \lambda V_\omega)) \in \mathcal{B}(\ell^2(\mathfrak{L})). \tag{2.4}$$

Here, $\Delta_{\mathrm{d}} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ is (up to a minus sign) the usual d-dimensional discrete Laplacian:

$$[\Delta_{\mathrm{d}}(\psi)](x) := 2d\psi(x) - \sum_{z\in\mathfrak{L},\ |z|=1} \psi(x+z), \quad x \in \mathfrak{L},\ \psi \in \ell^2(\mathfrak{L}). \tag{2.5}$$

In particular, for an independent identically distributed (i.i.d.) random potential V_ω, $(\Delta_{\mathrm{d}} + \lambda V_\omega)$ is the Anderson tight-binding model acting on the Hilbert space $\ell^2(\mathfrak{L})$. (Note that we could add some constant (chemical) potential to the discrete Laplacian without changing our proofs.)

For all $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, the condition

$$\tau_t^{(\omega,\lambda)}(a(\psi)) = a\big((\mathrm{U}_t^{(\omega,\lambda)})^* \psi\big), \quad t \in \mathbb{R},\ \psi \in \ell^2(\mathfrak{L}), \tag{2.6}$$

uniquely defines a family $\tau^{(\omega,\lambda)} := \{\tau_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ of (Bogoliubov) automorphisms of $\mathcal{U}$; see [7, theorem 5.2.5]. The one-parameter group $\tau^{(\omega,\lambda)}$ is strongly continuous and we denote its (unbounded) generator by $\delta^{(\omega,\lambda)}$.

2.3 Electromagnetic Fields

The electromagnetic potential is defined by a compactly supported time-dependent vector potential $\mathbf{A} \in \mathbf{C}_0^\infty$, where

$$\mathbf{C}_0^\infty := \bigcup_{l\in\mathbb{R}^+} \big\{\mathbf{A} : \mathbb{R}\times\mathbb{R}^d \mapsto (\mathbb{R}^d)^* \mid \exists \mathbf{B} \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d; (\mathbb{R}^d)^*) \text{ with } \mathbf{A}(t,x) = \mathbf{B}(t,x)\mathbf{1}[x \in [-l, l^d]]\big\}.$$

Here, $(\mathbb{R}^d)^*$ is the set of 1-forms[2] on $\mathbb{R}^d$ that take values in $\mathbb{R}$. In other words, as $[-l,l]^d$ is a compact subset of $\mathbb{R}^d$, $\mathbf{C}_0^\infty$ is the union

$$\mathbf{C}_0^\infty = \bigcup_{l\in\mathbb{R}^+} C_0^\infty(\mathbb{R}\times[-l,l]^d; (\mathbb{R}^d)^*)$$

[2] In a strict sense, one should take the dual space of the tangent spaces $T(\mathbb{R}^d)_x$, $x \in \mathbb{R}^d$.

of the space of smooth compactly supported functions $\mathbb{R} \times [-l, l]^d \to (\mathbb{R}^d)^*$ for $l \in \mathbb{R}^+$. The smoothness of $\mathbf{A}$ is not really necessary at this stage but will be technically convenient in subsequent papers. Here, only the continuous differentiability of the map $t \mapsto \mathbf{A}(t, \cdot)$ is really crucial to define below the electric field and the nonautonomous dynamics.

Since $\mathbf{A} \in \mathbf{C}_0^\infty$, $\mathbf{A}(t, x) = 0$ for all $t \leq t_0$, where $t_0 \in \mathbb{R}$ is some initial time. We use the Weyl gauge (also named temporal gauge) for the electromagnetic field and as a consequence,

$$E_{\mathbf{A}}(t, x) := -\partial_t \mathbf{A}(t, x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^d, \tag{2.7}$$

is the electric field associated with $\mathbf{A}$.

Note that the time $t_1 \geq t_0$ when the electric field is turned off can be chosen as arbitrarily large and one recovers the DC regime by taking $t_1 \gg 1$. However, for electric fields slowly varying in time, charge carriers have time to move and significantly change the charge density, producing an additional, self-generated, internal electric field. This contribution is not taken into account in our model.

Finally, observe that space-dependent electromagnetic potentials imply magnetic fields that interact with fermion spins. We neglect this contribution because such a term will become negligible for electromagnetic potentials slowly varying in space. This justifies the assumption of fermions with zero spin. In any case, our study can be performed for nonzero fermion spins exactly in the same way. We omit this generalization for simplicity.

2.4 Dynamics in the Presence of Time-Dependent Electromagnetic Fields

Recall that we only consider *negatively* charged fermions. We choose units such that the charge of fermions is -1. The (minimal) coupling of the vector potential $\mathbf{A} \in \mathbf{C}_0^\infty$ to the fermion system is achieved through a redefinition of the discrete Laplacian. Indeed, we define the time-dependent self-adjoint operator $\Delta_{\mathrm{d}}^{(\mathbf{A})} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ by

$$\left\langle \mathfrak{e}_x, \Delta_{\mathrm{d}}^{(\mathbf{A})} \mathfrak{e}_y \right\rangle = \exp\left(-i \int_0^1 [\mathbf{A}(t, \alpha y + (1-\alpha)x)](y - x) \mathrm{d}\alpha \right) \langle \mathfrak{e}_x, \Delta_{\mathrm{d}} \mathfrak{e}_y \rangle \tag{2.8}$$

for all $x, y \in \mathfrak{L}$, where $\langle \cdot, \cdot \rangle$ is here the canonical scalar product in $\ell^2(\mathfrak{L})$ and $\{\mathfrak{e}_x\}_{x \in \mathfrak{L}}$ is the canonical orthonormal basis $\mathfrak{e}_x(y) \equiv \delta_{x,y}$ of $\ell^2(\mathfrak{L})$. In Equation (2.8), $\alpha y + (1-\alpha)x$ and $y - x$ are seen as vectors in $\mathbb{R}^d$.

Observe that there is $l_0 \in \mathbb{R}^+$ such that

$$\Delta_{\mathrm{d}}^{(\mathbf{A})} - \Delta_{\mathrm{d}} \in \mathcal{B}(\ell^2([-l_0, l_0]^d \cap \mathfrak{L})) \subset \mathcal{B}(\ell^2(\mathfrak{L}))$$

for all times $t \in \mathbb{R}$, because $\mathbf{A}$ is by definition compactly supported. Note also that, for simplicity, the time dependence is often omitted in the notation,

$$\Delta_{\mathrm{d}}^{(\mathbf{A})} \equiv \Delta_{\mathrm{d}}^{(\mathbf{A}(t,\cdot))}, \quad t \in \mathbb{R},$$

but one has to keep in mind that the dynamics is *nonautonomous*.

Indeed, the Schrödinger equation on the one-particle Hilbert space $\ell^2(\mathfrak{L})$ with time-dependent Hamiltonian $(\Delta_{\mathrm{d}}^{(\mathbf{A})} + \lambda V_\omega)$ and initial value $\psi \in \ell^2(\mathfrak{L})$ at $t = t_0$ has a unique solution $\mathrm{U}_{t,t_0}^{(\omega,\lambda,\mathbf{A})}\psi$ for any $t \geq t_0$. Here, for any $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\mathbf{A} \in \mathbf{C}_0^\infty$,

$$\{\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s} \subset \mathcal{B}(\ell^2(\mathfrak{L}))$$

is the two-parameter group of unitary operators on $\ell^2(\mathfrak{L})$ generated by the (anti-self-adjoint) operator $-i(\Delta_{\mathrm{d}}^{(\mathbf{A})} + \lambda V_\omega)$:

$$\text{(2.9)}\quad \forall s,t \in \mathbb{R},\ t \geq s:\quad \partial_t \mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})} = -i(\Delta_{\mathrm{d}}^{(\mathbf{A}(t,\cdot))} + \lambda V_\omega)\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})},\qquad \mathrm{U}_{s,s}^{(\omega,\lambda,\mathbf{A})} := \mathbf{1}.$$

Since the map

$$\text{(2.10)}\qquad t \mapsto \left(\Delta_{\mathrm{d}}^{(\mathbf{A}(t,\cdot))} + \lambda V_\omega\right) \in \mathcal{B}(\ell^2(\mathfrak{L}))$$

from $\mathbb{R}$ to the space $\mathcal{B}(\ell^2(\mathfrak{L}))$ of bounded operators acting on $\ell^2(\mathfrak{L})$ is continuously differentiable for every $\mathbf{A} \in \mathbf{C}_0^\infty$, $\{\mathrm{U}_{t,s}^{(\omega)}\}_{t\geq s}$ is a norm-continuous two-parameter group of unitary operators. For more details, see Section 5.2.

Therefore, for all $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\mathbf{A} \in \mathbf{C}_0^\infty$, the condition

$$\text{(2.11)}\qquad \tau_{t,s}^{(\omega,\lambda,\mathbf{A})}(a(\psi)) = a\left(\left(\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}\right)^*\psi\right),\quad t \geq s,\ \psi \in \ell^2(\mathfrak{L}),$$

uniquely defines a family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ of Bogoliubov automorphisms of the C^*-algebra $\mathcal{U}$; see [7, theorem 5.2.5]. It is a strongly continuous two-parameter family that obeys the nonautonomous evolution equation

$$\text{(2.12)}\quad \forall s,t \in \mathbb{R},\ t \geq s:\quad \partial_t \tau_{t,s}^{(\omega,\lambda,\mathbf{A})} = \tau_{t,s}^{(\omega,\lambda,\mathbf{A})} \circ \delta_t^{(\omega,\lambda,\mathbf{A})},\quad \tau_{s,s}^{(\omega,\lambda,\mathbf{A})} := \mathbf{1},$$

with $\mathbf{1}$ being the identity of $\mathcal{U}$. Here, at any *fixed* time $t \in \mathbb{R}$, $\delta_t^{(\omega,\lambda,\mathbf{A})}$ is the infinitesimal generator of the (Bogoliubov) group $\{\tau_s^{(\omega,\lambda,\mathbf{A})}\}_{s\in\mathbb{R}} \equiv \{\tau_s^{(\omega,\lambda,\mathbf{A}(t,\cdot))}\}_{s\in\mathbb{R}}$ of automorphisms defined by replacing Δ_{d} with $\Delta_{\mathrm{d}}^{(\mathbf{A})}$ in (2.4); see (5.12). For more details on the properties of $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$, see also Sections 5.2–5.3.

Observe that one can equivalently use either (2.11) or (2.12) to define the dynamics; see also Proposition 5.4. However, only the second formulation (2.12) is appropriate to study transport properties of systems of interacting fermions on the lattice in its algebraic formulation.

Remark 2.1 (Heisenberg Picture). The initial value problem (2.12) can easily be understood in the Heisenberg picture. The time evolution of any observable $B_s \in \mathcal{B}(\ell^2(\mathfrak{L}))$ at initial time $t = s \in \mathbb{R}$ equals $B_t = (\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})})^* B_s \mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}$ for $t \geq s$, which yields

$$\forall t \geq s:\quad \partial_t B_t = \left(\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}\right)^* i\left[\Delta_{\mathrm{d}}^{(\mathbf{A})} + \lambda V_\omega, B_s\right]\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}.$$

The action of the symmetric derivation $\delta_t^{(\omega,\lambda,\mathbf{A})}$ in (2.12) is related to the above commutator, whereas the map $B \mapsto (\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})})^* B \mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}$ leads to the family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ in the second quantization; see also Theorem 5.3.

2.5 Time-Dependent State of the System

States on the C^*-algebra $\mathcal{U}$ are, by definition, continuous linear functionals $\rho \in \mathcal{U}^*$ that are normalized and positive, i.e., $\rho(\mathbf{1}) = 1$ and $\rho(A^*A) \geq 0$ for all $A \in \mathcal{U}$.

It is well-known that, at finite volume, the thermodynamic equilibrium of the system is described by the corresponding Gibbs state, which is the unique state minimizing the free energy. It is stationary and satisfies the so-called KMS condition. The latter also makes sense in infinite volume and is thus used to define the thermodynamic equilibrium of the infinite system; see, e.g., Section A.2, in particular Theorem A.3.

Therefore, we assume that, for any function $\omega \in \Omega$ and strength $\lambda \in \mathbb{R}_0^+$ of the static potential, the state of the system before the electric field is switched on is the unique $(\tau^{(\omega,\lambda)}, \beta)$-KMS state $\varrho^{(\beta,\omega,\lambda)}$; see [7, example 5.3.2.] or [23, theorem 5.9]. Here, $\beta \in \mathbb{R}^+$ (i.e., $\beta > 0$) is the inverse temperature of the fermion system at equilibrium.

Since $\mathbf{A}(t,x) = 0$ for all $t \leq t_0$, the time evolution of the state of the system thus equals

$$\rho_t^{(\beta,\omega,\lambda,\mathbf{A})} := \begin{cases} \varrho^{(\beta,\omega,\lambda)}, & t \leq t_0, \\ \varrho^{(\beta,\omega,\lambda)} \circ \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A})}, & t \geq t_0. \end{cases} \tag{2.13}$$

Note that the definition does not depend on the particular choice of initial time t_0 because of the stationarity of the KMS state $\varrho^{(\beta,\omega,\lambda)}$ with respect to the unperturbed dynamics (cf. (5.3)). The state $\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}$ is, by construction, a quasi-free state.

3 Heat Production

3.1 Heat Production as Quantum Relative Entropy

Joule's law describes the rate at which resistance converts electric energy into *heat*. That quantity of heat is not characterized here by a *local* increase of temperature, but it is proportional to an *entropy* production. The proportionality coefficient is of course the temperature $\beta^{-1} \in \mathbb{R}^+$, which is seen as a *global* parameter of the *infinite* system. The heat production is thus, by definition, a relative quantity with respect to the reference state of the system, that is, the thermal (or equilibrium) state $\varrho^{(\beta,\omega,\lambda)}$ for $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, and $\lambda \in \mathbb{R}_0^+$. Its mathematical formulation requires Araki's notion of *relative entropy* [2, 3, 22].

The latter takes a simple form for finite-dimensional C^*-algebras like the local fermion algebras $\{\mathcal{U}_\Lambda\}_{\Lambda\in\mathcal{P}_f(\mathfrak{L})}$: Let $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ and denote by tr the normalized trace on $\mathcal{U}_\Lambda$, also named the tracial state of $\mathcal{U}_\Lambda$. By [4, lemma 3.1 (i)], for any state $\rho \in \mathcal{U}_\Lambda^*$, there is a unique adjusted density matrix $\mathrm{d}_\rho \in \mathcal{U}$, that is, $\mathrm{d}_\rho \geq 0$,

$\mathrm{tr}(\mathrm{d}_\rho) = 1$, and $\rho(A) = \mathrm{tr}(\mathrm{d}_\rho A)$ for all $A \in \mathcal{U}_\Lambda$. We define by $\mathrm{supp}(\rho)$ the smallest projection $\mathrm{P} \in \mathcal{U}_\Lambda$ such that $\rho(\mathrm{P}) = 1$. Then the relative entropy of a state $\rho_1 \in \mathcal{U}_\Lambda^*$ with respect to $\rho_2 \in \mathcal{U}_\Lambda^*$ is defined by (A.9) for $\mathcal{X} = \mathcal{U}_\Lambda$ and, by finite dimensionality, it equals

$$S_{\mathcal{U}_\Lambda}(\rho_1|\rho_2) = \begin{cases} \rho_1(\ln \mathrm{d}_{\rho_1} - \ln \mathrm{d}_{\rho_2}) \in \mathbb{R}_0^+, & \text{if } \mathrm{supp}(\rho_2) \geq \mathrm{supp}(\rho_1), \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.1}$$

under the convention $x \ln x|_{x=0} := 0$; see Lemma A.1. It is always a nonnegative quantity; see, for instance, [22, eq. (1.3) and prop. 1.1].

For more general C^*-algebras like the CAR C^*-algebra $\mathcal{U}$ of the infinite system, Araki's definition of relative entropy [2, 3, 22] invokes the modular theory. This definition is rather abstract, albeit standard, and for the reader's convenience we thus postpone it until Section A.1. Indeed, using the boxes

$$\Lambda_L := \{(x_1, \ldots, x_d) \in \mathfrak{L} : |x_1|, \ldots, |x_d| \leq L\} \in \mathcal{P}_f(\mathfrak{L}) \tag{3.2}$$

for any $L \in \mathbb{R}^+$, we observe that $\{\mathcal{U}_{\Lambda_L}\}_{L\in\mathbb{R}^+}$ is an increasing net of C^*-subalgebras of the C^*-algebra $\mathcal{U}$. Moreover, the $*$-algebra

$$\mathcal{U}_0 := \bigcup_{L\in\mathbb{R}^+} \mathcal{U}_{\Lambda_L} \subset \mathcal{U} \tag{3.3}$$

of local elements is, by construction, dense in $\mathcal{U}$. (Indeed, $\mathcal{U}$ is by definition the completion of the normed $*$-algebra $\mathcal{U}_0$.) We thus define the relative entropy of any state $\rho_1 \in \mathcal{U}^*$ with respect to $\rho_2 \in \mathcal{U}^*$ by

$$\begin{aligned} \mathrm{S}(\rho_1|\rho_2) &:= \lim_{L\to\infty} S_{\mathcal{U}_{\Lambda_L}}(\rho_{1,\Lambda_L}|\rho_{2,\Lambda_L}) \\ &= \sup_{L\in\mathbb{R}^+} S_{\mathcal{U}_{\Lambda_L}}(\rho_{1,\Lambda_L}|\rho_{2,\Lambda_L}) \in [0,\infty] \end{aligned} \tag{3.4}$$

with ρ_{1,Λ_L} and ρ_{2,Λ_L} being the restrictions to $\mathcal{U}_{\Lambda_L}$ of the states ρ_1 and ρ_2, respectively. By [22, prop. 5.23 (vi)], this limit exists and equals Araki's relative entropy, i.e., $\mathrm{S}(\rho_1|\rho_2) = S_{\mathcal{U}}(\rho_1|\rho_2)$ with $S_{\mathcal{U}}$ defined by (A.9) for $\mathcal{X} = \mathcal{U}$. In particular, it is a nonnegative (possibly infinite) quantity. Since $\mathrm{S} = S_{\mathcal{U}}$, note that the second equality in (3.4) follows from [22, prop. 5.23 (iv)], which in turn results from the Uhlmann monotonicity theorem for Schwarz mappings [22, prop. 5.3].

Therefore, the *heat production* is defined from (2.13) and (3.4) as follows:

DEFINITION 3.1 (Heat Production). For any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\mathbf{A} \in \mathbf{C}_0^\infty$, $\mathbf{Q}^{(\omega,\mathbf{A})} \equiv \mathbf{Q}^{(\beta,\omega,\lambda,\mathbf{A})}$ is defined as a map from $\mathbb{R}$ to $\overline{\mathbb{R}}$ by

$$\mathbf{Q}^{(\omega,\mathbf{A})}(t) := \beta^{-1}\mathrm{S}\big(\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\big|\varrho^{(\beta,\omega,\lambda)}\big) \in [0,\infty].$$

The heat production $\mathbf{Q}^{(\omega,\mathbf{A})}(t)$ may a priori be infinite for some time $t \in \mathbb{R}$. We prove in the next section that $\mathbf{Q}^{(\omega,\mathbf{A})}$ is finite for all times. In particular, the states $\varrho^{(\beta,\omega,\lambda)}$ and $\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}$ are globally similar.

3.2 Heat Production and First Law of Thermodynamics

> *In a thermodynamic process of a closed system, the increment in the internal energy is equal to the difference between the increment of heat accumulated by the system and the increment of work done by it.*
>
> Clausius (English translation), 1850

This is the celebrated *first law of thermodynamics*; see [9]. For a historical and mathematical account on thermodynamics, see, e.g., [13]. See also [1] for an interesting derivation of this law from quantum statistical mechanics.

In the system considered here, the increment of *total* energy follows from the interaction between electromagnetic fields and charged fermions. Part of this increment results from the change of the internal state of fermions. It is interpreted below as an increment of *internal* energy of the system. The other part is an electromagnetic energy that is generally nonvanishing even if the internal state of fermions would stay at equilibrium. For this reason, this part is seen below as an increase of electromagnetic *potential* energy of charged particles within the electromagnetic field. As the system under consideration does not interact with surroundings and thus can neither perform work nor exchange heat, all the increase of internal energy is expected to be converted into heat by the first law of thermodynamics. Therefore, the heat production $\mathbf{Q}^{(\omega,\mathbf{A})}$ should be related to the increment of the internal energy of the system. This is far from being explicit in Definition 3.1. We show that it is indeed the case for the fermion system considered here.

To this end, we first need to give precise definitions of the increments of *total*, *internal*, and (electromagnetic) *potential* energies. In quantum mechanics, these energies should be associated with total, internal, and potential energy observables, that is in our case, self-adjoint elements of $\mathcal{U}$. They are defined as follows: For any $L \in \mathbb{R}^+$, $[L] \in \mathbb{N}$ being its integer part, the *internal* energy observable in the box Λ_L (3.2) of side length $2[L]+1$ is defined by

$$H_L^{(\omega,\lambda)} := \sum_{x,y\in\Lambda_L} \langle \mathfrak{e}_x, (\Delta_{\mathrm{d}} + \lambda V_\omega)\mathfrak{e}_y \rangle a_x^* a_y \in \mathcal{U}. \tag{3.5}$$

It is the second quantization of the one-particle operator $\Delta_{\mathrm{d}} + \lambda V_\omega$ restricted to the subspace $\ell^2(\Lambda_L) \subset \ell^2(\mathfrak{L})$. When the electromagnetic field is switched on, i.e., for $t \geq t_0$, the (time-dependent) *total* energy observable in the box Λ_L is then equal to $H_L^{(\omega,\lambda)} + W_t^{\mathbf{A}}$, where, for any $\mathbf{A} \in \mathbf{C}_0^\infty$ and $t \in \mathbb{R}$,

$$W_t^{\mathbf{A}} := \sum_{x,y\in\Lambda_L} \left\langle \mathfrak{e}_x, \left(\Delta_{\mathrm{d}}^{(\mathbf{A})} - \Delta_{\mathrm{d}}\right)\mathfrak{e}_y \right\rangle a_x^* a_y \in \mathcal{U} \tag{3.6}$$

is the electromagnetic *potential* energy observable.

As a consequence, for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \in \mathbb{R}$, the *total* energy increment engendered by the interaction with the electromagnetic field equals

$$\lim_{L\to\infty}\left\{\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\left(H_L^{(\omega,\lambda)}+W_t^{\mathbf{A}}\right)-\varrho^{(\beta,\omega,\lambda)}\left(H_L^{(\omega,\lambda)}\right)\right\}=\mathbf{S}^{(\omega,\mathbf{A})}(t)+\mathbf{P}^{(\omega,\mathbf{A})}(t). \tag{3.7}$$

Here, $\mathbf{S}^{(\omega,\mathbf{A})}\equiv\mathbf{S}^{(\beta,\omega,\lambda,\mathbf{A})}$ is the *internal* energy increment defined as a map from $\mathbb{R}$ to $\overline{\mathbb{R}}$ by

$$\mathbf{S}^{(\omega,\mathbf{A})}(t):=\lim_{L\to\infty}\left\{\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\left(H_L^{(\omega,\lambda)}\right)-\varrho^{(\beta,\omega,\lambda)}\left(H_L^{(\omega,\lambda)}\right)\right\}, \tag{3.8}$$

whereas the electromagnetic *potential* energy (increment) $\mathbf{P}^{(\omega,\mathbf{A})}\equiv\mathbf{P}^{(\beta,\omega,\lambda,\mathbf{A})}$ is defined as a map from $\mathbb{R}$ to $\mathbb{R}$ by

$$\mathbf{P}^{(\omega,\mathbf{A})}(t):=\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\left(W_t^{\mathbf{A}}\right)=\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\left(W_t^{\mathbf{A}}\right)-\varrho^{(\beta,\omega,\lambda)}(W_{t_0}^{\mathbf{A}}) \tag{3.9}$$

for any $\beta\in\mathbb{R}^+$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_0^+$, and $\mathbf{A}\in\mathbf{C}_0^\infty$. In particular, $\mathbf{S}^{(\omega,\mathbf{A})}$ is only nonvanishing if the state of the fermion system changes, whereas $\mathbf{P}^{(\omega,\mathbf{A})}$ vanishes in the absence of external electromagnetic potential.

Note that

$$\mathbf{P}^{(\omega,\mathbf{A})}(t)=\left\{\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\left(W_t^{\mathbf{A}}\right)-\varrho^{(\beta,\omega,\lambda)}\left(W_t^{\mathbf{A}}\right)\right\}+\varrho^{(\beta,\omega,\lambda)}\left(W_t^{\mathbf{A}}\right). \tag{3.10}$$

The last part is the raw electromagnetic energy given to the system at equilibrium. It is the so-called diamagnetic energy, which will be studied in subsequent papers. The energy increment between brackets in the right-hand side of (3.10) will also be analyzed in detail later and is part of a so-called paramagnetic energy increment. It is the amount of electromagnetic potential energy absorbed or released by the fermion system to change its internal state.

It is not a priori obvious that the limits (3.7) and (3.8) exist because, in general,

$$\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\left(H_L^{(\omega,\lambda)}\right)=\mathcal{O}(L^d).$$

We show below that these limits have nevertheless finite real values. Indeed, we infer from Theorem 5.8 that, for any $\beta\in\mathbb{R}^+$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_0^+$, and $\mathbf{A}\in\mathbf{C}_0^\infty$, the energy sum (3.7) is the *work* performed on the system by the electromagnetic field at time $t\geq t_0$:

$$\mathbf{S}^{(\omega,\mathbf{A})}(t)+\mathbf{P}^{(\omega,\mathbf{A})}(t)=\int_{t_0}^{t}\rho_s^{(\beta,\omega,\lambda,\mathbf{A})}\left(\partial_s W_s^{\mathbf{A}}\right)\mathrm{d}s. \tag{3.11}$$

Here, $\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(\partial_t W_t^{\mathbf{A}})$ is interpreted as the infinitesimal work of the electromagnetic field at time $t\in\mathbb{R}$. See, for instance, discussions in [7, sec. 5.4.4.]. Note that this conservation law is not completely obvious in our case because the considered system is *infinitely extended*.

We now derive the first law of thermodynamics:

THEOREM 3.2 (First Law of Thermodynamics). *For any $\beta \in \mathbb{R}^{+}$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_{0}^{+}$, $\mathbf{A} \in \mathbf{C}_{0}^{\infty}$, and $t \in \mathbb{R}$,*

$$\mathbf{Q}^{(\omega,\mathbf{A})}(t) = \mathbf{S}^{(\omega,\mathbf{A})}(t) \in \mathbb{R}_{0}^{+}.$$

In particular, the maps $\mathbf{Q}^{(\omega,\mathbf{A})}$ and $\mathbf{S}^{(\omega,\mathbf{A})}$ defined by Definition 3.1 *and* (3.8)*, respectively, always take positive and finite values for all times.*

PROOF. All arguments are given in Section 5.4; see Theorem 5.5 and Corollaries 5.6–5.7. Note also that, by definition,

$$\mathbf{P}^{(\omega,\mathbf{A})}(t) = \mathbf{S}^{(\omega,\mathbf{A})}(t) = \mathbf{Q}^{(\omega,\mathbf{A})}(t) = 0$$

whenever $t \leq t_0$. □

Observe that the state $\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}$ of the fermion system still evolves for $t \geq t_1$ when the electromagnetic field is turned off. Indeed, for any $\beta \in \mathbb{R}^{+}$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_{0}^{+}$, $\mathbf{A} \in \mathbf{C}_{0}^{\infty}$, and $t \geq t_1$,

$$\rho_t^{(\beta,\omega,\lambda,\mathbf{A})} = \rho_{t_1}^{(\beta,\omega,\lambda,\mathbf{A})} \circ \tau_{t-t_1}^{(\omega,\lambda,\mathbf{A})}.$$

Despite that, the total heat created by the electromagnetic field stays *constant* as soon as the electromagnetic field is turned off: By Theorem 3.2, $\mathbf{S}^{(\omega,\mathbf{A})}$ is the heat production due to the interaction with the electromagnetic field and from (3.11) we deduce that, for all $\beta \in \mathbb{R}^{+}$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_{0}^{+}$, $\mathbf{A} \in \mathbf{C}_{0}^{\infty}$, and $t \geq t_1$,

$$\mathbf{Q}^{(\omega,\mathbf{A})}(t) = \mathbf{S}^{(\omega,\mathbf{A})}(t) = \int_{t_0}^{t_1} \rho_s^{(\beta,\omega,\lambda,\mathbf{A})}\big(\partial_s W_s^{\mathbf{A}}\big)\mathrm{d}s = \mathbf{S}^{(\omega,\mathbf{A})}(t_1) = \mathbf{Q}^{(\omega,\mathbf{A})}(t_1).$$

If

$$\mathbf{Q}^{(\omega,\mathbf{A})}(t) = \int_{t_0}^{t_1} \rho_s^{(\beta,\omega,\lambda,\mathbf{A})}\big(\partial_s W_s^{\mathbf{A}}\big)\mathrm{d}s > 0$$

for any $t \geq t_1$, a strictly positive amount of electromagnetic work is absorbed by the infinite-volume fermion system. We will show in a subsequent paper that this situation (almost surely) appears for $\lambda > 0$, as expected from Joule's law.

For specific static potentials V_ω like constant ones, the heat conduction in the infinite system still implies a dissipation of energy, or thermalization, in the sense that, for any *fixed* $L \in \mathbb{R}^{+}$,

$$\lim_{t\to\infty}\big\{\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\big(H_L^{(\omega,\lambda)}\big) - \varrho^{(\beta,\omega,\lambda)}\big(H_L^{(\omega,\lambda)}\big)\big\} = 0. \tag{3.12}$$

The latter can be verified by explicit computations. Beside the special case of constant potentials V_ω, the situation is more complicated. Indeed, the self-adjoint operator $\Delta_{\mathrm{d}} + \lambda V_\omega$ acting on $\ell^2(\mathfrak{L})$ can have eigenvalues. In particular, the energy $\mathbf{Q}^{(\omega,\mathbf{A})}(t_1)$ for $t \geq t_1$ could be stored in bound states, in contrast with the perfect conducting case (3.12). As a consequence, we can only hope for an asymptotic version of the above result:

$$\limsup_{\lambda\to 0}\lim_{t\to\infty}\big\{\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\big(H_L^{(\omega,\lambda)}\big) - \varrho^{(\beta,\omega,\lambda)}\big(H_L^{(\omega,\lambda)}\big)\big\} = 0$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and each $L \in \mathbb{R}^+$.

Remark 3.3 (Internal Energies). The internal energy as defined in [1, eq. (15)] rather corresponds in our case to the total energy increment. Then, (3.11) is, in Salem-Fröhlich's interpretation, the expression of the first law of thermodynamics. Indeed, we have a closed system that cannot exchange heat energy with its surrounding like in [1, eq. (16)]. In their viewpoint, $\mathbf{P}^{(\omega,\mathbf{A})}$ should be seen as a Helmholtz free energy, i.e., the available energy that can perform work. In fact, the authors in [1, eq. (16)] focus on the heat exchanged with the surrounding, whereas we do not consider it and concentrate our study on the heat production within the fermion system.

3.3 Heat Production at Small Electromagnetic Fields

The physical situation we will use to investigate Joule's and Ohm's laws is as follows: We start with a macroscopic bulk containing conducting fermions. This is idealized by taking an infinite system of noninteracting fermions as explained above. Then, the heat production or the conductivity is measured in a region that is very small with respect to the size of the bulk, but very large with respect to the lattice spacing of the crystal.

We implement this hierarchy of space scales by rescaling vector potentials. That means, for any $l \in \mathbb{R}^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$, we consider the space-rescaled vector potential

$$\mathbf{A}_l(t,x) := \mathbf{A}(t, l^{-1}x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^d. \tag{3.13}$$

Then, to ensure that a macroscopic number of lattice sites is involved, we eventually perform the limit $l \to \infty$. Indeed, the scaling factor l^{-1} used in (3.13) means, at fixed l, that the space scale of the electric field (2.7) is infinitesimal with respect to the macroscopic bulk (which is the whole space), whereas the lattice spacing gets infinitesimal with respect to the space scale of the vector potential when $l \to \infty$.

Furthermore, Ohm's law is a linear response to electric fields. Therefore, we also rescale the strength of the electromagnetic potential $\mathbf{A}_l$ by a real parameter $\eta \in \mathbb{R}$ and will eventually take the limit $\eta \to 0$ in a subsequent paper.

In the limit $(\eta, l^{-1}) \to (0,0)$, it turns out that the heat production $\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}$ and the internal energy increment $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}$, respectively, defined by Definition 3.1 and (3.8), are of order $\mathcal{O}(\eta^2 l^d)$. This can be understood in a physical sense by the fact that the energy contained in the electromagnetic field, that is, its L^2-norm, is also of order $\mathcal{O}(\eta^2 l^d)$, by classical electrodynamics. Then, in order to get Joule's and Ohm's laws, we need to give an explicit expression for the term of order $\mathcal{O}(\eta^2 l^d)$ of $\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}$, uniformly with respect to some parameters. This is performed in Section 5.5 by using two important tools, also used several times in subsequent papers:

- A Dyson-Phillips expansion in terms of multicommutators of the strongly continuous two-parameter family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ defined by (2.11); see Section 5.2.

- Tree-decay bounds on multicommutators as explained in Section 4.

Recall that multicommutators are defined by induction as follows:

$$[B_1, B_2]^{(2)} := [B_1, B_2] := B_1 B_2 - B_2 B_1, \quad B_1, B_2 \in \mathcal{U}, \tag{3.14}$$

and, for all integers $k > 2$,

$$[B_1, B_2, \ldots, B_{k+1}]^{(k+1)} := [B_1, [B_2, \ldots, B_{k+1}]^{(k)}], \quad B_1, \ldots, B_{k+1} \in \mathcal{U}. \tag{3.15}$$

In fact, provided $\eta \in \mathbb{R}$ is sufficiently small, we get in Section 5.5 a representation of $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}$ as a power series in η such that all k-order terms in η are of order $\mathcal{O}(l^d)$, as $l \to \infty$, i.e., they behave as the volume of the support of the electromagnetic field.

THEOREM 3.4 (Heat Production at Small Fields). *Let* $\mathbf{A} \in \mathbf{C}_0^\infty$. *Then the heat production has the following properties*:

(i) *Multicommutator series. There exists* $\eta_0 \equiv \eta_{0,\mathbf{A}} \in \mathbb{R}^+$ *such that, for all* $|\eta| \in [0, \eta_0]$, $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, *and* $t \geq t_0$,

$$\begin{aligned}\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t) &= \sum_{k\in\mathbb{N}} \sum_{x,z\in\mathfrak{L},|z|\leq 1} i^k \langle \mathfrak{e}_x, (\Delta_{\mathrm{d}} + \lambda V_\omega)\mathfrak{e}_{x+z}\rangle \int_{t_0}^{t} \mathrm{d}s_1 \cdots \int_{t_0}^{s_{k-1}} \mathrm{d}s_k \\ &\quad \varrho^{(\beta,\omega,\lambda)}\big(\big[W^{\eta\mathbf{A}_l}_{s_k-t_0,s_k}, \ldots, W^{\eta\mathbf{A}_l}_{s_1-t_0,s_1}, \tau^{(\omega,\lambda)}_{t-t_0}(a^*_x a_{x+z})\big]^{(k+1)}\big)\end{aligned} \tag{3.16}$$

with $W^{\eta\mathbf{A}_l}_{t,s} := \tau^{(\omega,\lambda)}_t(W^{\eta\mathbf{A}_l}_s) \in \mathcal{U}$ *for any* $t, s \in \mathbb{R}$. *The above sum is absolutely convergent.*

(ii) *Uniform analyticity at* $\eta = 0$. *The function* $\eta \mapsto \mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}$ *is real analytic on* $\mathbb{R}$ *and there exist* $\eta_1 \equiv \eta_{1,\mathbf{A}} \in \mathbb{R}^+$ *and* $D \equiv D_{\mathbf{A}} \in \mathbb{R}^+$ *such that, for all* $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \geq t_0$, *and* $m \in \mathbb{N}$,

$$\big|\partial_\eta^m \mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t)\big|_{\eta=0}\big| \leq D l^d \big(\eta_1^{-m} m!\big). \tag{3.17}$$

In particular, the Taylor series in η *of* $l^{-d}\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}$ *is absolutely convergent in a neighborhood of* $\eta = 0$, *uniformly in the parameters* $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ *and* $t \geq t_0$.

PROOF. To prove (i), combine Theorem 3.2 with Equation (5.43). See also Lemma 5.10. The second assertion (ii) is a direct consequence of Corollary 5.9 and Lemma 5.11 together with Theorem 3.2. Note that Lemma 5.11 shows slightly stronger bounds than (3.17). □

Note that $\mathbf{Q}^{(\omega,0)}(t) = 0$ and thus, (3.11) directly gives the estimate

$$\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t) - \mathbf{Q}^{(\omega,0)}(t) = \mathcal{O}(|\eta| l^d)$$

for the rest of order one of the Taylor expansion of $\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}$. This is a special case of Theorem 3.4(ii), which implies, for all $M \in \mathbb{N}$ and $\eta \in [0, \eta_1]$, that

$$\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t) - \sum_{m=1}^{M} \frac{\eta^m}{m!} \big(\partial_\eta^m \mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t)\big|_{\eta=0}\big) = \mathcal{O}(|\eta|^{M+1} l^d). \tag{3.18}$$

By explicit computations, the Taylor coefficients of order zero and one of the function $\eta \mapsto \mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t)$ always vanish. Hence, using Theorem 3.4(ii), one shows that

$$l^{-d}\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t) = \mathcal{O}(\eta^2) + \mathcal{O}(|\eta|^3). \tag{3.19}$$

The term $\mathcal{O}(\eta^2)$ can be made explicit, whereas the correction term of order $\mathcal{O}(\eta^3)$ is uniformly bounded in $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $t \geq t_0$. The detailed analysis of the leading term $\mathcal{O}(\eta^2)$ is postponed to a subsequent paper.

As a consequence, for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \in \mathbb{R}$, one can analyze the density $\mathbf{q} \equiv \mathbf{q}^{(\beta,\omega,\lambda,\mathbf{A})}$ of heat production by the limits

$$\begin{aligned}\mathbf{q}(t) &:= \lim_{(\eta,l^{-1})\to(0,0)} \{(\eta^2 l^d)^{-1} \mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t)\} \\ &= \lim_{(\eta,l^{-1})\to(0,0)} \{(\eta^2 l^d)^{-1} \mathbf{S}^{(\omega,\eta\mathbf{A}_l)}(t)\} := \mathbf{s}(t);\end{aligned}$$

see Theorem 3.2. This study will lead to Joule's law, which describes the rate at which resistance in the fermion system converts electric energy into heat energy. The details of such a study, for instance the existence of the above limits, are the subject of a companion paper.

By (3.18), the density of heat production should be a real analytic function at $\eta = 0$. Hence, Theorem 3.4 also makes possible the study of nonquadratic and nonlinear corrections to Joule's law and Ohm's law, respectively.

4 Tree-Decay Bounds

Note that

$$W_{t,s}^{\eta\mathbf{A}_l} := \tau_t^{(\omega,\lambda)}(W_s^{\eta\mathbf{A}_l}) = \mathcal{O}(|\eta| l^d)$$

for any $t, s \in \mathbb{R}$ and $\mathbf{A} \in \mathbf{C}_0^\infty$; see also (3.13). Thus, using Equation (3.16), naive bounds on its right-hand side predict that, for some constant $D > 1$,

$$\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t) = \mathcal{O}\big(D^{|\eta| l^d}\big).$$

To obtain the much more accurate estimate

$$\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}(t) = \mathcal{O}(\eta^2 l^d) \tag{4.1}$$

and to prove Theorem 3.4, we need good bounds on the multicommutators in the series (3.16). This is achieved by using the so-called tree-decay bounds on the expectation of such multicommutators. Indeed, tree-decay bounds we derive here are a useful tool to control multicommutators of products of annihilation and creation operators. This technique will also be used many times in subsequent papers in order to derive Joule's and Ohm's laws.

Observe that (4.1) implies thermodynamic behavior of the heat production with respect to $l \in \mathbb{R}^+$; i.e., $\mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}$ is proportional to the volume l^d. This kind of issue is well-known in statistical physics of interacting systems where cluster or graph expansions are used to obtain such a behavior for quantities like the free-energy or the ground-state energy at large volumes. In the langage of constructive physics, the main result of the present section, that is, Corollary 4.3, yields the convergence of a tree expansion for the heat production.

The proof of Corollary 4.3 uses Theorem 4.1 as an important ingredient. The latter is a tree expansion for multicommutators of monomials in annihilation and creation operators. Such a kind of a combinatorial result was already used before, for instance in [16]. In fact, Theorem 4.1 is very similar to arguments used in [16, sec. 4].

Before going into details, let us first illustrate what will be proven in Theorem 4.1. The aim is to simplify N-fold multicommutators of monomials in annihilation and creation operators, as for example

$$[a^*(\psi_1)a(\psi_2)a^*(\psi_3)a^*(\psi_4), a^*(\psi_5)a(\psi_6), \ldots]^{(N)} \tag{4.2}$$

with $\psi_1, \psi_2, \ldots \in \ell^2(\mathfrak{L})$. See (3.14)–(3.15) for the precise definition of multicommutators. At a first glance one expects sums over monomials containing all occurring annihilation and creation operators. Because of the structure of the multicommutator, there are certain terms that can be summed up, getting then monomials containing all occurring annihilation and creation operators except two, times the anticommutator of those two; see (4.8). This is useful because the anticommutator is a multiple of the identity, cf. (2.1). This procedure can be iterated $N-1$ times in order to reduce the number of annihilation and creation operators in the remaining monomials. As one might expect, only pairs of creation and annihilation operators that come from *different* entries of the multicommutator can be removed. This is why we consider in the following a family of trees, similar to [16]. The $N-1$ edges (bonds) of those trees (containing N vertices) represent the contractions of annihilation and creation operators into anticommutators. The vertices of such trees stand for the N entries of the N-fold multicommutator.

Now we need to introduce some notation to express the monomials in annihilation and creation operators in a convenient way, before formulating Theorem 4.1. Each of the entries of the N-fold multicommutator is a product of annihilation and creation operators, which we characterize by certain finite index sets $\overline{\Lambda}_1, \Lambda_1, \ldots, \overline{\Lambda}_N, \Lambda_N \subset \mathbb{N}$, where the set $\overline{\Lambda}_i$ refers to creation operators in entry i and Λ_i to annihilation operators in the same entry. For example, we choose for (4.2) the sets

$$\overline{\Lambda}_1 = \{1,3,4\}, \quad \Lambda_1 = \{2\}, \quad \overline{\Lambda}_2 = \{5\}, \quad \Lambda_2 = \{6\}, \quad \ldots. \tag{4.3}$$

The kind of products we are interested in allows us to restrict our considerations to index sets $\overline{\Lambda}_1, \Lambda_1, \ldots, \overline{\Lambda}_N, \Lambda_N \subset \mathbb{N}$ that are nonempty, mutually disjoint and such that

$$|\overline{\Lambda}_j| + |\Lambda_j| := 2n_j \in 2\mathbb{N}$$

for all $j \in \{1, \ldots, N\}$. Hence, each entry in the multicommutator contains an even number of annihilation and creation operators. To simplify the notation, we set

$$\Omega_j := (\{+\} \times \overline{\Lambda}_j) \cup (\{-\} \times \Lambda_j)$$

for all $j \in \{1, \ldots, N\}$. To determine the position of annihilation and creation operators in the monomial of the j^{th} entry, we choose a numbering of Ω_j, that is, a bijective map

$$\pi_j : \{1, \ldots, 2n_j\} \to \Omega_j. \tag{4.4}$$

In the example (4.2)–(4.3),

$$\Omega_1 = \{(+, 1), (+, 3), (+, 4), (-, 2)\},$$

and its numbering is defined by

$$\pi_1(1) = (+, 1), \quad \pi_1(2) = (-, 2), \quad \pi_1(3) = (+, 3), \quad \pi_1(4) = (+, 4).$$

Furthermore, for all $x \in \bigcup_{j=1}^{N} \overline{\Lambda}_j \cup \Lambda_j$, let $\psi_x \in \ell^2(\mathfrak{L})$ be the corresponding wave function and denote (only in this section) the annihilation and creation operators, respectively, by

$$a(-, x) := a(\psi_x) \quad \text{and} \quad a(+, x) := a^*(\psi_x).$$

Using this notation, we then define the monomials

$$\mathfrak{p}_j := \prod_{k=1}^{2n_j} a(\pi_j(k)) \tag{4.5}$$

in $a(\pm, x)$ for all $j \in \{1, \ldots, N\}$. Recall that $\mathfrak{p}_j$ is the j^{th} entry in the N-fold multicommutator.

To formulate Theorem 4.1, we need two more things. Recall that a tree is a connected graph that has no loops. Here we have a finite number of labeled vertices, denoted by $1, \ldots, N$, and (nonoriented) bonds between these vertices. For example, the bond connecting vertices i and j is denoted by $\{i, j\} = \{j, i\}$. A tree is characterized by the set of its $N - 1$ bonds. The family of trees we use is defined as follows: Let $\mathcal{T}_2$ be the set of all trees with exactly two vertices. This set contains a unique tree $T = \{\{1, 2\}\}$, which in turn contains the unique bond $\{1, 2\}$, i.e., $\mathcal{T}_2 := \{\{\{1, 2\}\}\}$. Then, for each integer $N \geq 3$, we recursively define the set $\mathcal{T}_N$ of trees with N vertices by

$$\mathcal{T}_N := \big\{\{\{k, N\}\} \cup T : k = 1, \ldots, N - 1,\ T \in \mathcal{T}_{N-1}\big\}. \tag{4.6}$$

In other words, $\mathcal{T}_N$ is the set of all trees with vertex set $\mathcal{V}_N := \{1, \ldots N\}$ for which $N \in \mathcal{V}_N$ is a leaf, and if the leaf N is removed, the vertex $N - 1$ is a leaf in the remaining tree and so on.

Now, for every tree $T \in \mathcal{T}_N$, we define maps $\mathbf{x}, \mathbf{y} : T \to \bigcup_{j=1}^{N} \Omega_j$ that choose, for each bond $\{i, j\} \in T$, a point in the set Ω_i and one point in the set Ω_j, respectively. More precisely, we assume for $i < j$ that $\mathbf{x}(\{i, j\}) \in \Omega_i$ and $\mathbf{y}(\{i, j\}) \in \Omega_j$. The induced orientation of the bond is completely arbitrary because of the symmetry of anticommutators. The set of all those maps is given by

$$\mathcal{K}_T := \left\{ (\mathbf{x}, \mathbf{y}) \,\middle|\, x, \mathbf{y} : T \to \bigcup_{j=1}^{N} \Omega_j \right.$$
$$\left. \text{with } \mathbf{x}(b) \in \Omega_i,\ \mathbf{y}(b) \in \Omega_j \text{ for } b = \{i, j\} \in T,\ i < j \right\}.$$

We are finally ready to express an N-fold multicommutator of products of annihilation and creation operators as a sum over trees $T \in \mathcal{T}_N$ of monomials in annihilation and creation operators:

THEOREM 4.1 (Multicommutators as Sums over Trees). *Let $N \geq 2$. Then, for all $T \in \mathcal{T}_N$ and $(\mathbf{x}, \mathbf{y}) \in \mathcal{K}_T$, there are constants*

$$\mathrm{m}_T(\mathbf{x}, \mathbf{y}) \in \{-1, 0, 1\}$$

and injective maps

$$\pi_T(\mathbf{x}, \mathbf{y}) : \{1, 2, \ldots, 2\overline{\mathrm{N}}\} \to \bigcup_{j=1}^{N} \Omega_j \setminus (\mathbf{x}(T) \cup \mathbf{y}(T))$$

where $\overline{\mathrm{N}} := \sum_{j=1}^{N} n_j - (N-1) \geq 1$ such that

$$[\mathfrak{p}_N, \ldots, \mathfrak{p}_1]^{(N)} = \sum_{T \in \mathcal{T}_N} \sum_{(\mathbf{x},\mathbf{y}) \in \mathcal{K}_T} \mathrm{m}_T(\mathbf{x}, \mathbf{y}) \mathfrak{p}_T(\mathbf{x}, \mathbf{y}) \prod_{b \in T} \{a(\mathbf{x}(b)), a(\mathbf{y}(b))\}, \tag{4.7}$$

with $\{B_1, B_2\} := B_1 B_2 + B_2 B_1$ being the anticommutator of $B_1, B_2 \in \mathcal{U}$ and

$$\mathfrak{p}_T(\mathbf{x}, \mathbf{y}) := \prod_{k=1}^{2\overline{\mathrm{N}}} a(\pi_T(\mathbf{x}, \mathbf{y})(k)).$$

PROOF. We first observe that, for any integers $n_1, n_2 \in \mathbb{N}$ and all elements $B_1, \ldots, B_{2n_2} \in \mathcal{U}$ and $\widetilde{B}_1, \ldots, \widetilde{B}_{2n_1} \in \mathcal{U}$,

$$\begin{aligned} &[B_1 \cdots B_{2n_2}, \widetilde{B}_1 \cdots \widetilde{B}_{2n_1}] \\ &\quad = \sum_{\substack{1 \leq k_1 \leq 2n_1 \\ 1 \leq k_2 \leq 2n_2}} (-1)^{k_1+1} B_1 \cdots B_{k_2-1} \widetilde{B}_1 \cdots \widetilde{B}_{k_1-1} \\ &\qquad \times \{B_{k_2}, \widetilde{B}_{k_1}\} \widetilde{B}_{k_1+1} \cdots \widetilde{B}_{2n_1} B_{k_2+1} \cdots B_{2n_2}, \end{aligned} \tag{4.8}$$

see [16, eq. (4.18)]. Note also that, for $k_2 = 1$, one obtains

$$(-1)^{k_1+1} B_1 \cdots B_{1-1} \widetilde{B}_1 \cdots \widetilde{B}_{k_1-1} \{B_1, \widetilde{B}_{k_1}\} \widetilde{B}_{k_1+1} \cdots \widetilde{B}_{2n_1} B_2 \cdots B_{2n_2}.$$

This of course has to be understood as

$$(-1)^{k_1+1} \widetilde{B}_1 \cdots \widetilde{B}_{k_1-1} \{B_1, \widetilde{B}_{k_1}\} \widetilde{B}_{k_1+1} \cdots \widetilde{B}_{2n_1} B_2 \cdots B_{2n_2}.$$

Similar remarks can be made for the cases $k_1 = 1, 2n_1$ and $k_2 = 2n_2$. We now prove the assertion by induction.

For $N = 2$, the set $\mathcal{T}_2 := \{\{\{1,2\}\}\}$ consists of only one tree $T = \{\{1,2\}\}$. Using (4.5) and (4.8) we get

$$\begin{aligned}[\mathfrak{p}_2, \mathfrak{p}_1] = \sum_{\substack{1 \le k_1 \le 2n_1 \\ 1 \le k_2 \le 2n_2}} & (-1)^{k_1+1} a(\pi_2(1)) \cdots a(\pi_2(k_2-1)) a(\pi_1(1)) \cdots a(\pi_1(k_1-1)) \\ & \times \{a(\pi_2(k_2)), a(\pi_1(k_1))\} a(\pi_1(k_1+1)) \cdots a(\pi_1(2n_1)) \\ & \times a(\pi_2(k_2+1)) \cdots a(\pi_2(2n_2)).\end{aligned} \tag{4.9}$$

Note that $\{a(\pi_2(k_2)), a(\pi_1(k_1))\}$ is always a multiple of the identity in $\mathcal{U}$; see (2.1) and (2.2). Therefore, the assertion for $N = 2$ directly follows from the previous equality by observing that the sum over k_1 and k_2 in (4.9) corresponds to the sum over $(\mathbf{x}, \mathbf{y}) \in \mathcal{K}_{\{\{1,2\}\}}$ in (4.7) by choosing

$$\begin{aligned}&\mathfrak{p}_{\{\{1,2\}\}}(\mathbf{x}, \mathbf{y}) \\ &\quad := a(\pi_2(1)) \cdots a(\pi_2(k_2-1)) a(\pi_1(1)) \cdots a(\pi_1(k_1-1)) \\ &\qquad \times\ a(\pi_1(k_1+1)) \cdots a(\pi_1(2n_1)) a(\pi_2(k_2+1)) \cdots a(\pi_2(2n_2))\end{aligned} \tag{4.10}$$

for

$$\begin{aligned}\mathbf{x}(\{1,2\}) &= \pi_1(k_1) \in \Omega_1, \quad k_1 \in \{1, \ldots, 2n_1\}, \\ \mathbf{y}(\{1,2\}) &= \pi_2(k_2) \in \Omega_2, \quad k_2 \in \{1, \ldots, 2n_2\}.\end{aligned}$$

For $(\mathbf{x}, \mathbf{y}) \in \mathcal{K}_{\{\{1,2\}\}}$ as above, the constant $\mathrm{m}_{\{\{1,2\}\}}(\mathbf{x}, \mathbf{y})$ equals $(-1)^{k_1+1} \in \{-1, 1\} \subset \{-1, 0, 1\}$, whereas the associated map

$$\pi_{\{\{1,2\}\}}(\mathbf{x}, \mathbf{y}) : \{1, 2, \ldots, 2\overline{\mathrm{N}}\} \to \Omega_1 \cup \Omega_2 \setminus (\mathbf{x}(\{\{1,2\}\}) \cup \mathbf{y}(\{\{1,2\}\}))$$

with

$$\overline{\mathrm{N}} := (n_1 + n_2) - 1 \ge 1$$

depends on the order of the factors in the right-hand side of (4.10):

$$\pi_{\{\{1,2\}\}}(\mathbf{x}, \mathbf{y})(k) := \begin{cases} \pi_2(k), & k \in \{1, 2, \ldots, k_2 - 1\}. \\ \pi_1(k - k_2 + 1), & k \in \{k_2, \ldots, k_2 + k_1 - 2\}. \\ \pi_1(k - k_2 + 2), & k \in \{k_2 + k_1 - 1, \ldots, 2n_1 - 2 + k_2\}. \\ \pi_2(k - 2n_1 + 2), & k \in \{2n_1 - 2 + k_2 + 1, \ldots, 2\overline{\mathrm{N}}\}. \end{cases}$$

We assume now that the assertion holds for some fixed integer $N \geq 2$. Recall that N-fold multicommutators are defined by (3.14)–(3.15). In particular,

$$[\mathfrak{p}_{N+1}, \ldots, \mathfrak{p}_1]^{(N+1)} = [\mathfrak{p}_{N+1}, [\mathfrak{p}_N, \ldots, \mathfrak{p}_1]^{(N)}]$$

where, by assumption,

$$[\mathfrak{p}_N, \ldots, \mathfrak{p}_1]^{(N)} = \sum_{T \in \mathcal{T}_N} \sum_{(\mathbf{x},\mathbf{y}) \in \mathcal{K}_T} \mathrm{m}_T(\mathbf{x}, \mathbf{y}) \mathfrak{p}_T(\mathbf{x}, \mathbf{y}) \prod_{b \in T} \{a(\mathbf{x}(b)), a(\mathbf{y}(b))\},$$

as stated in the theorem. Therefore,

$$\begin{aligned} [\mathfrak{p}_{N+1}, \ldots, \mathfrak{p}_1]^{(N+1)} = \sum_{T \in \mathcal{T}_N} \sum_{(\mathbf{x},\mathbf{y}) \in \mathcal{K}_T} & \mathrm{m}_T(\mathbf{x}, \mathbf{y}) [\mathfrak{p}_{N+1}, \mathfrak{p}_T(\mathbf{x}, \mathbf{y})] \\ & \times \prod_{b \in T} \{a(\mathbf{x}(b)), a(\mathbf{y}(b))\}, \end{aligned} \tag{4.11}$$

whereas, using again (4.8),

$$\begin{aligned} &[\mathfrak{p}_{N+1}, \mathfrak{p}_T(\mathbf{x}, \mathbf{y})] \\ &\quad = \sum_{\substack{1 \leq k_1 \leq 2\overline{\mathrm{N}} \\ 1 \leq k_2 \leq 2n_{N+1}}} (-1)^{k_1+1} a(\pi_{N+1}(1)) \cdots a(\pi_{N+1}(k_2 - 1)) \\ &\qquad \times a(\pi_T(1)) \cdots a(\pi_T(k_1 - 1)) \\ &\qquad \times a(\pi_T(k_1 + 1)) \cdots a(\pi_T(2\overline{\mathrm{N}})) \\ &\qquad \times a(\pi_{N+1}(k_2 + 1)) \cdots a(\pi_{N+1}(2n_{N+1})) \\ &\qquad \times \{a(\pi_{N+1}(k_2)), a(\pi_T(k_1))\}. \end{aligned} \tag{4.12}$$

Note that, for simplicity, we sometimes use (as above) the notation $\pi_T \equiv \pi_T(\mathbf{x}, \mathbf{y})$.

To get now the assertion for $(N + 1)$-fold multicommutators, for any $(\mathbf{x}, \mathbf{y}) \in \mathcal{K}_T$, we define:

$$\begin{aligned} X &:= \pi_T(k_1) \in \bigcup_{j=1}^{N} \Omega_j \setminus (\mathbf{x}(T) \cup \mathbf{y}(T)), \quad && k_1 \in \{1, \ldots, 2\overline{\mathrm{N}}\}, \\ Y &:= \pi_{N+1}(k_2) \in \Omega_{N+1}, && k_2 \in \{1, \ldots, 2n_{N+1}\}, \end{aligned}$$

as well as

$$\tilde{\mathrm{m}}_T(X, Y) := (-1)^{k_1+1}$$

and

$$\begin{aligned} \tilde{\mathfrak{p}}_T(\mathbf{x}, \mathbf{y}, X, Y) := {} & a(\pi_{N+1}(1)) \cdots a(\pi_{N+1}(k_2 - 1)) \\ & \times a(\pi_T(1)) \cdots a(\pi_T(k_1 - 1)) a(\pi_T(k_1 + 1)) \cdots a(\pi_T(2\overline{\mathrm{N}})) \\ & \times a(\pi_{N+1}(k_2 + 1)) \cdots a(\pi_{N+1}(2n_{N+1})). \end{aligned}$$

Then, by (4.11)-(4.12), one has

$$[\mathfrak{p}_{N+1}, \ldots, \mathfrak{p}_1]^{(N+1)} = \sum_{T \in \mathcal{T}_N} \sum_{(\mathbf{x},\mathbf{y}) \in \mathcal{K}_T} \sum_{X \in (\Omega_1 \cup \cdots \cup \Omega_N) \backslash (\mathbf{x}(T) \cup \mathbf{y}(T))} \sum_{Y \in \Omega_{N+1}} \mathrm{m}_T(\mathbf{x},\mathbf{y}) \widetilde{\mathrm{m}}_T(X,Y) \widetilde{\mathfrak{p}}_T(\mathbf{x},\mathbf{y},X,Y) \{a(X), a(Y)\} \prod_{b \in T} \{a(\mathbf{x}(b)), a(\mathbf{y}(b))\}.$$

This last equation can clearly be rewritten as

$$\begin{aligned} &[\mathfrak{p}_{N+1}, \ldots, \mathfrak{p}_1]^{(N+1)} \\ &\quad = \sum_{T \in \mathcal{T}_N} \sum_{(\mathbf{x},\mathbf{y}) \in \mathcal{K}_T} \sum_{k \in \{1,\ldots,N\}} \sum_{X_{\{k,N+1\}} \in \Omega_k} \sum_{Y_{\{k,N+1\}} \in \Omega_{N+1}} \\ &\quad \mathbf{1}[X_{\{k,N+1\}} \notin (\mathbf{x}(T) \cup \mathbf{y}(T))] \mathrm{m}_T(\mathbf{x},\mathbf{y}) \widetilde{\mathrm{m}}_T(X_{\{k,N+1\}}, Y_{\{k,N+1\}}) \\ &\quad \times \widetilde{\mathfrak{p}}_T(\mathbf{x},\mathbf{y},X_{\{k,N+1\}}, Y_{\{k,N+1\}}) \\ &\quad \times \{a(X_{\{k,N+1\}}), a(Y_{\{k,N+1\}})\} \prod_{b \in T} \{a(\mathbf{x}(b)), a(\mathbf{y}(b))\}. \end{aligned} \tag{4.13}$$

Since Ω_j, $j \in \{1, \ldots, N\}$, are, by definition, mutually disjoint sets, the latter yields the assertion for the $(N+1)$-fold multicommutator. Indeed, one only needs to define, for any tree $T \in \mathcal{T}_{N+1}$ with $N+1$ vertices and fixed $(\mathbf{x},\mathbf{y}) \in \mathcal{K}_T$, an appropriate constant $\mathrm{m}_T(\mathbf{x},\mathbf{y}) \in \{-1, 0, 1\}$ and map $\pi_T(\mathbf{x},\mathbf{y})$. This can directly be deduced from (4.6) and (4.13); we omit the details. □

Because of (4.13) note that, for any $N \geq 2$, all $T \in \mathcal{T}_N$ and $(\mathbf{x},\mathbf{y}) \in \mathcal{K}_T$, the constants $\mathrm{m}_T(\mathbf{x},\mathbf{y})$ of Theorem 4.1 satisfy $\mathrm{m}_T(\mathbf{x},\mathbf{y}) = 0$ whenever

$$|\mathbf{x}(T)| + |\mathbf{y}(T)| < 2(N-1).$$

Similar to $\{\pi_j\}_{j \in \{1,\ldots,N\}}$ (see (4.4)), the maps $\pi_T(\mathbf{x},\mathbf{y})$ are (injective) numberings:

$$\{x : \pi_T(\mathbf{x},\mathbf{y})(k) = (+, x) \text{ for } k \in \{1, \ldots, 2\overline{\mathrm{N}}\}\} = \bigcup_{j=1}^{N} \overline{\Lambda}_j \setminus \overline{\Lambda}_{\mathbf{x},\mathbf{y}},$$

$$\{x : \pi_T(\mathbf{x},\mathbf{y})(k) = (-, x) \text{ for } k \in \{1, \ldots, 2\overline{\mathrm{N}}\}\} = \bigcup_{j=1}^{N} \Lambda_j \setminus \Lambda_{\mathbf{x},\mathbf{y}},$$

where, for any $T \in \mathcal{T}_N$ and $(\mathbf{x},\mathbf{y}) \in \mathcal{K}_T$,

$$\Lambda_{\mathbf{x},\mathbf{y}} := \{z \in \mathfrak{L} : (-, z) \in \{\mathbf{x}(b), \mathbf{y}(b)\} \text{ for some } b \in T\},$$

$$\overline{\Lambda}_{\mathbf{x},\mathbf{y}} := \{z \in \mathfrak{L} : (+, z) \in \{\mathbf{x}(b), \mathbf{y}(b)\} \text{ for some } b \in T\}.$$

We conclude this section by the notion of tree-decay bounds: Let $\rho \in \mathcal{U}^*$ be any state and $\tau \equiv \{\tau_t\}_{t \in \mathbb{R}}$ be any one-parameter group of automorphisms on the C^*-algebra $\mathcal{U}$. We say that (ρ, τ) satisfies *tree-decay bounds* with parameters $\epsilon \in \mathbb{R}^+$

and $t_0, t \in \mathbb{R}$, $t_0 < t$, if there is a finite constant $D \in \mathbb{R}^+$ such that, for any integer $N \geq 2$, $s_1, \ldots, s_N \in [t_0, t]$, $x_1, \ldots, x_N \in \mathfrak{L}$, and all $z_1, \ldots, z_N \in \mathfrak{L}$ satisfying $|z_i| = 1$ for $i \in \{1, \ldots, N\}$,

$$|\rho([\tau_{s_1}(a^*_{x_1} a_{x_1+z_1}), \ldots, \tau_{s_N}(a^*_{x_N} a_{x_N+z_N})]^{(N)})| \leq D^{N-1} \mathbf{v}_N^{(\epsilon)}(x_1, \ldots, x_N), \tag{4.14}$$

where

$$\mathbf{v}_N^{(\epsilon)}(x_1, \ldots, x_N) = \sum_{T \in \mathcal{T}_N} \prod_{\{k,l\} \in T} \frac{1}{1 + |x_k - x_l|^{d+\epsilon}}, \quad x_1, \ldots, x_N \in \mathfrak{L}.$$

(Recall that $\mathfrak{L} := \mathbb{Z}^d$ with $d \in \mathbb{N}$.)

Such a property is used in Section 5.5 and will be exploited many times in the subsequent papers for $\tau = \tau^{(\omega,\lambda)}$ and $\rho = \varrho^{(\beta,\omega,\lambda)}$ with $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, and $\lambda \in \mathbb{R}_0^+$. In fact, using Theorem 4.1 we show below that the one-parameter Bogoliubov group $\tau^{(\omega,\lambda)}$ of automorphisms defined by (2.6) and any state ρ satisfy tree-decay bounds. Indeed, observe first the following elementary lemma:

LEMMA 4.2 (Correlation Decays). *For any $T, \epsilon \in \mathbb{R}^+$, there is a finite constant $D \in \mathbb{R}^+$ such that*

$$|\langle \mathfrak{e}_x, \mathrm{e}^{it(\Delta_{\mathrm{d}} + \lambda V_\omega)} \mathfrak{e}_y \rangle| \leq \frac{D}{1 + |x - y|^{d+\epsilon}}$$

for all $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in [-T, T]$, and $x, y \in \mathfrak{L}$. Recall that $\{\mathfrak{e}_x\}_{x \in \mathfrak{L}}$ is the canonical orthonormal basis of $\ell^2(\mathfrak{L})$ defined by $\mathfrak{e}_x(y) \equiv \delta_{x,y}$ for all $x, y \in \mathfrak{L}$.

PROOF. Let $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$, and $x, y \in \mathfrak{L}$. Using the Trotter-Kato formula and the canonical orthonormal basis $\{\mathfrak{e}_x\}_{x \in \mathfrak{L}}$ of $\ell^2(\mathfrak{L})$, we first observe that

$$\begin{aligned} &\langle \mathfrak{e}_x, \mathrm{e}^{it(\Delta_{\mathrm{d}} + \lambda V_\omega)} \mathfrak{e}_y \rangle \\ &\quad = \lim_{m \to \infty} \left\langle \mathfrak{e}_x, \left[\mathrm{e}^{\frac{it}{m}\Delta_{\mathrm{d}}} \mathrm{e}^{\frac{it}{m}\lambda V_\omega} \right]^m \mathfrak{e}_y \right\rangle \\ &\quad = \lim_{m \to \infty} \lim_{L \to \infty} \sum_{x_1, \ldots, x_{m-1} \in \Lambda_L} \left\langle \mathfrak{e}_x, \mathrm{e}^{\frac{it}{m}\Delta_{\mathrm{d}}} \mathfrak{e}_{x_1} \right\rangle \cdots \left\langle \mathfrak{e}_{x_{m-1}}, \mathrm{e}^{\frac{it}{m}\Delta_{\mathrm{d}}} \mathfrak{e}_y \right\rangle \\ &\qquad \times \mathrm{e}^{\frac{it}{m}\lambda V_\omega(x_1)} \times \cdots \times \mathrm{e}^{\frac{it}{m}\lambda V_\omega(y)}, \end{aligned} \tag{4.15}$$

where Λ_L is the finite box (3.2) of side length $2[L] + 1$ for $L \in \mathbb{R}^+$. Writing now the exponential $\mathrm{e}^{\frac{it}{m}\Delta_{\mathrm{d}}}$ as a power series and using the definition (2.5) of the discrete Laplacian Δ_{d}, we arrive at the upper bound

$$\left| \left\langle \mathfrak{e}_x, \mathrm{e}^{\frac{it}{m}\Delta_{\mathrm{d}}} \mathfrak{e}_y \right\rangle \right| \leq \mathrm{e}^{\frac{4d|t|}{m}} \left\langle \mathfrak{e}_x, \mathrm{e}^{-\frac{|t|}{m}\Delta_{\mathrm{d}}} \mathfrak{e}_y \right\rangle, \quad x, y \in \mathfrak{L}, \ t \in \mathbb{R}, \ m \in \mathbb{N}. \tag{4.16}$$

Therefore, we infer from (4.15)–(4.16) that

$$\left|\left\langle \mathfrak{e}_x, \mathrm{e}^{it(\Delta_\mathrm{d}+\lambda V_\omega)}\mathfrak{e}_y\right\rangle\right| \leq \mathrm{e}^{4d|t|}\left\langle \mathfrak{e}_x, \mathrm{e}^{-|t|\Delta_\mathrm{d}}\mathfrak{e}_y\right\rangle. \tag{4.17}$$

Note that Δ_d is explicitly given in Fourier space by the dispersion relation

$$E(p) := 2[d - (\cos(p_1) + \cdots + \cos(p_d))], \quad p \in [-\pi, \pi]^d.$$

Thus, explicit computations show that, for all $s \in \mathbb{R}$,

$$\langle \mathfrak{e}_x, \mathrm{e}^{s\Delta_\mathrm{d}}\mathfrak{e}_y\rangle = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{e}^{sE(p)-ip\cdot(x-y)}\mathrm{d}^d p,$$

which, combined with (4.17), implies the assertion. □

By (2.1) and (2.6),

$$\left\|\{\tau_{s_1}^{(\omega,\lambda)}(a_x^*), \tau_{s_2}^{(\omega,\lambda)}(a_y)\}\right\| = \left|\left\langle \mathfrak{e}_x, \mathrm{e}^{i(s_2-s_1)(\Delta_\mathrm{d}+\lambda V_\omega)}\mathfrak{e}_y\right\rangle\right| \tag{4.18}$$

for every $s_1, s_2 \in \mathbb{R}$, $x, y \in \mathfrak{L}$, $\omega \in \Omega$, and $\lambda \in \mathbb{R}_0^+$. Hence, for any $\epsilon \in \mathbb{R}^+$ and $t_0, t \in \mathbb{R}$, $t_0 < t$, we infer from Lemma 4.2 the existence of a finite constant $D \in \mathbb{R}^+$ (depending only on ϵ, t_0, t) such that

$$\left\|\left\{\tau_{s_1}^{(\omega,\lambda)}(a_x^*), \tau_{s_2}^{(\omega,\lambda)}(a_y)\right\}\right\| \leq \frac{D}{1+|x-y|^{d+\epsilon}} \tag{4.19}$$

for all $s_1, s_2 \in [t_0, t]$, $x, y \in \mathfrak{L}$, $\omega \in \Omega$, and $\lambda \in \mathbb{R}_0^+$. Using this and Theorem 4.1, we obtain (4.14) with a uniform constant $D < \infty$ not depending on $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$:

COROLLARY 4.3 (Uniform Tree-Decay Bounds). *Let ρ be any arbitrary state on $\mathcal{U}$ and $\tau = \tau^{(\omega,\lambda)}$ be the one-parameter Bogoliubov group of automorphisms defined by* (2.6) *for $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$. Then, for every $\epsilon \in \mathbb{R}^+$ and $t_0, t \in \mathbb{R}$, $t_0 < t$, there is $D = D_{\epsilon,t_0,t} \in \mathbb{R}^+$ such that the tree-decay bound* (4.14) *holds for all $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$.*

PROOF. Choose in Theorem 4.1 sets $\bar{\Lambda}_j, \Lambda_j$ containing exactly one element and note that, in this case, $|\mathcal{K}_T| = 2^{2|T|} = 2^{2(N-1)}$. Observe also that $\|\mathfrak{p}_T(\mathbf{x}, \mathbf{y})\| \leq 1$ as the corresponding vectors ψ_x have norm 1. The assertion then follows from (4.19) and Theorem 4.1. □

5 Proofs of Main Results

5.1 Preliminary

For the reader's convenience we start by reviewing a few important definitions and some standard mathematical results used in our proofs.

Recall that $\mathfrak{L} := \mathbb{Z}^d$ with $d \in \mathbb{N}$, and $\mathcal{P}_f(\mathfrak{L}) \subset 2^{\mathfrak{L}}$ is the set of all finite subsets of $\mathfrak{L}$. For any $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $\mathcal{U}_\Lambda$ is the CAR C^*-algebra generated by the identity $\mathbf{1}$ and the annihilation operators $\{a_x\}_{x\in\Lambda}$. It is isomorphic to the finite-dimensional C^*-algebra $\mathcal{B}(\bigwedge \mathcal{H}_\Lambda)$ of all linear operators on the fermion Fock space $\bigwedge \mathcal{H}_\Lambda$, where $\mathcal{H}_\Lambda := \bigoplus_{x\in\Lambda} \mathcal{H}_x$ is the Cartesian product of copies $\mathcal{H}_x$, $x \in \Lambda$, of the one-dimensional Hilbert space $\mathcal{H} \equiv \mathbb{C}$ (i.e., the one-particle Hilbert space $\mathcal{H}_\Lambda$ is isomorphic to $\mathbb{C}^\Lambda$). The CAR C^*-algebra $\mathcal{U}$ is the (separable) C^*-algebra defined by the inductive limit of $\{\mathcal{U}_\Lambda\}_{\Lambda\in\mathcal{P}_f(\mathfrak{L})}$. Note here that $\mathcal{U}_{\Lambda'} \subset \mathcal{U}_\Lambda$ whenever $\Lambda' \subset \Lambda$. For any one-particle wave function $\psi \in \ell^2(\mathfrak{L})$ we define annihilation and creation operators $a(\psi), a^*(\psi) \in \mathcal{U}$ of a (spinless) fermion; see (2.2).

For $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, the unperturbed dynamics of the fermion system studied here is given by the one-parameter group $\tau^{(\omega,\lambda)} := \{\tau_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ of Bogoliubov automorphisms on the algebra $\mathcal{U}$ uniquely defined by the condition (2.6), that is,

$$\tau_t^{(\omega,\lambda)}(a(\psi)) = a(\mathrm{e}^{it(\Delta_\mathrm{d}+\lambda V_\omega)}\psi), \quad t \in \mathbb{R},\ \psi \in \ell^2(\mathfrak{L}); \tag{5.1}$$

see [7, theorem 5.2.5]. As $\tau_t^{(\omega,\lambda)}$ is an automorphism of $\mathcal{U}$, by definition, we have in particular that

$$\tau_t^{(\omega,\lambda)}(B_1B_2) = \tau_t^{(\omega,\lambda)}(B_1)\tau_t^{(\omega,\lambda)}(B_2), \quad B_1, B_2 \in \mathcal{U},\ t \in \mathbb{R}. \tag{5.2}$$

Physically, (5.1) means that the fermionic particles do not experience any mutual force: They interact with each other via the Pauli exclusion principle only; i.e., they form an ideal lattice fermion system. From (2.3) and the norm-continuity of the unitary group $\{\mathrm{e}^{it(\Delta_\mathrm{d}+\lambda V_\omega)}\}_{t\in\mathbb{R}}$, it follows that the (Bogoliubov) group $\tau^{(\omega,\lambda)}$ of automorphisms is strongly continuous. $(\mathcal{U}, \tau^{(\omega,\lambda)})$ is thus a C^*-dynamical system.

For each $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, the generator of the strongly continuous group $\tau^{(\omega,\lambda)}$ is denoted by $\delta^{(\omega,\lambda)}$. It is a symmetric unbounded derivation. This means that the domain $\mathrm{Dom}(\delta^{(\omega,\lambda)})$ of $\delta^{(\omega,\lambda)}$ is a dense $*$-subalgebra of $\mathcal{U}$ and, for all $B_1, B_2 \in \mathrm{Dom}(\delta^{(\omega,\lambda)})$,

$$\delta^{(\omega,\lambda)}(B_1)^* = \delta^{(\omega,\lambda)}(B_1^*), \quad \delta^{(\omega,\lambda)}(B_1B_2) = \delta^{(\omega,\lambda)}(B_1)B_2 + B_1\delta^{(\omega,\lambda)}(B_2).$$

Recall that states on the C^*-algebra $\mathcal{U}$ are linear functionals $\rho \in \mathcal{U}^*$ that are normalized and positive, i.e., $\rho(\mathbf{1}) = 1$ and $\rho(A^*A) \geq 0$ for all $A \in \mathcal{U}$. Thermal equilibrium states of the fermion system under consideration can be defined, at inverse temperature $\beta \in \mathbb{R}^+$ and for any $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, through the bounded positive operator

$$\mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)} := \frac{1}{1+\mathrm{e}^{\beta(\Delta_\mathrm{d}+\lambda V_\omega)}} \in \mathcal{B}(\ell^2(\mathfrak{L})).$$

Indeed, the so-called *symbol* $\mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)}$ uniquely defines a (faithful) *quasi-free* state $\varrho^{(\beta,\omega,\lambda)}$ on the CAR algebra $\mathcal{U}$ by the conditions $\varrho^{(\beta,\omega,\lambda)}(\mathbf{1}) = 1$ and

$$\varrho^{(\beta,\omega,\lambda)}\big(a^*(f_1)\cdots a^*(f_m)a(g_n)\cdots a(g_1)\big) = \delta_{m,n}\det\big(\big[\langle g_k, \mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)} f_j\rangle\big]_{j,k}\big)$$

for all $\{f_j\}_{j=1}^m, \{g_j\}_{j=1}^n \subset \ell^2(\mathfrak{L})$ and $m, n \in \mathbb{N}$. $\langle\cdot,\cdot\rangle$ is here the scalar product in $\ell^2(\mathfrak{L})$.

The state $\varrho^{(\beta,\omega,\lambda)} \in \mathcal{U}^*$ is the unique $(\tau^{(\omega,\lambda)}, \beta)$-KMS state of the C^*-dynamical system $(\mathcal{U}, \tau^{(\omega,\lambda)})$. This means that, for every $B_1, B_2 \in \mathcal{U}$, the map

$$t \mapsto F_{B_1,B_2}(t) := \varrho^{(\beta,\omega,\lambda)}\big(B_1 \tau_t^{(\omega,\lambda)}(B_2)\big)$$

from $\mathbb{R}$ to $\mathbb{C}$ extends uniquely to a continuous map on $\mathbb{R} + i[0,\beta] \subset \mathbb{C}$, which is holomorphic on $\mathbb{R} + i(0,\beta)$, such that

$$F_{B_1,B_2}(t + i\beta) = \varrho^{(\beta,\omega,\lambda)}\big(\tau_t^{(\omega,\lambda)}(B_2)B_1\big)$$

for all $t \in \mathbb{R}$. The latter is named the *KMS condition* or the *modular condition* (when $\beta = 1$) in the context of von Neumann algebras.

The KMS condition is usually taken as the mathematical characterization of thermal equilibriums of C^*-dynamical systems. This definition of thermal equilibrium states for infinite systems is rather abstract. However, it can be physically motivated from a maximum entropy principle by observing that $\varrho^{(\beta,\omega,\lambda)}$ is the unique weak-$*$ limit of Gibbs states $\varrho^{(\beta,\omega,\lambda,L)}$ (A.17)–(A.18) as $L \to \infty$. See Theorem A.3. Moreover, KMS states are stationary, and thus $\varrho^{(\beta,\omega,\lambda)}$ is invariant under the dynamics defined by the (Bogoliubov) group $\tau^{(\omega,\lambda)}$ of automorphisms:

$$\varrho^{(\beta,\omega,\lambda)} \circ \tau_t^{(\omega,\lambda)} = \varrho^{(\beta,\omega,\lambda)}, \quad \beta \in \mathbb{R}^+,\ \omega \in \Omega,\ \lambda \in \mathbb{R}_0^+,\ t \in \mathbb{R}. \tag{5.3}$$

5.2 Series Representation of Dynamics

The assertions of this subsection are similar to [7, prop. 5.4.26.]. Note, however, that the generator $\delta^{(\omega,\lambda)}$ of the (unperturbed) one-parameter group $\tau^{(\omega,\lambda)}$ is an *unbounded* symmetric derivation, in contrast to [7, prop. 5.4.26.]. Here, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\mathbf{A} \in \mathbf{C}_0^\infty$ are arbitrarily fixed. See Sections 2.2–2.3.

We start our proofs by explicitly expressing the automorphism $\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}$ of $\mathcal{U}$ in terms of a series involving multicommutators. Meanwhile, we give an alternative characterization of the two-parameter family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ as a solution of an abstract Cauchy initial value problem. This last observation is very useful in order to generalize the present results to *interacting* fermion systems.

First, recall that there is a unique two-parameter group $\{\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ which is a norm-continuous solution of the nonautonomous Cauchy initial value problem (2.9), that is,

$$\forall s,t \in \mathbb{R},\ t \geq s\colon \quad \partial_t \mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})} = -i\left(\Delta_{\mathrm{d}}^{(\mathbf{A}(t,\cdot))} + \lambda V_\omega\right)\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}, \quad \mathrm{U}_{s,s}^{(\omega,\lambda,\mathbf{A})} := \mathbf{1}.$$

(The restriction $t \geq s$ is not essential here and $\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}$ could also be defined for all $s,t \in \mathbb{R}$.) Indeed, $\Delta_{\mathrm{d}} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ and the map

$$t \mapsto \mathbf{w}_t^{\mathbf{A}} := \big(\Delta_{\mathrm{d}}^{(\mathbf{A}(t,\cdot))} - \Delta_{\mathrm{d}}\big) \in \mathcal{B}(\ell^2(\mathfrak{L})) \tag{5.4}$$

from $\mathbb{R}$ to the set $\mathcal{B}(\ell^2(\mathfrak{L}))$ of bounded operators acting on $\ell^2(\mathfrak{L})$ is continuously differentiable for every $\mathbf{A} \in \mathbf{C}_0^\infty$. Hence, $\{\mathrm{U}_{t,s}^{(\omega)}\}_{t\geq s}$ can be explicitly written as the

Dyson-Phillips series

$$\begin{aligned} &\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})} - \mathrm{U}_{t-s}^{(\omega,\lambda)} = \\ &\quad \sum_{k\in\mathbb{N}} (-i)^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k \mathrm{U}_{t-s_1}^{(\omega,\lambda)} \mathbf{w}_{s_1}^{\mathbf{A}} \mathrm{U}_{s_1-s_2}^{(\omega,\lambda)} \cdots \mathrm{U}_{s_{k-1}-s_k}^{(\omega,\lambda)} \mathbf{w}_{s_k}^{\mathbf{A}} \mathrm{U}_{s_k-s}^{(\omega,\lambda)} \end{aligned} \tag{5.5}$$

for any $t \geq s$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\mathbf{A} \in \mathbf{C}_0^\infty$. Since all operators are bounded, it is easy to check that $\{\mathrm{U}_{t,s}^{(\omega)}\}_{t\geq s}$ is a family of unitary operators.

We are now in position to represent the Bogoliubov automorphisms $\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}$ defined by (2.11) as a Dyson-Phillips series involving the unperturbed dynamics defined by the one-parameter group $\tau^{(\omega,\lambda)} := \{\tau_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$; see (2.4) and (2.6). To this end, for every $\mathbf{A} \in \mathbf{C}_0^\infty$, we denote the second quantization of $\mathbf{w}_t^{\mathbf{A}}$ by

$$\begin{aligned} W_t^{\mathbf{A}} = \sum_{x,y\in\mathfrak{L}} &\left[\exp\left(-i\int_0^1 [\mathbf{A}(t,\alpha y + (1-\alpha)x)](y-x)\mathrm{d}\alpha\right) - 1\right] \\ &\times \langle \mathfrak{e}_x, \Delta_{\mathrm{d}}\mathfrak{e}_y\rangle a_x^* a_y; \end{aligned} \tag{5.6}$$

see (2.8), (3.6), and (5.4). Note that there is a finite subset $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ such that $W_t^{\mathbf{A}} \in \mathcal{U}_\Lambda$ for all $t \in \mathbb{R}$ because $\mathbf{A} \in \mathbf{C}_0^\infty$. We also define the continuously differentiable map

$$t \mapsto L_t^{\mathbf{A}} := i[W_t^{\mathbf{A}}, \cdot\,] \in \mathcal{B}(\mathcal{U}) \tag{5.7}$$

from $\mathbb{R}$ to the set $\mathcal{B}(\mathcal{U})$ of bounded operators acting on $\mathcal{U}$.

THEOREM 5.1 (Dynamics as a Dyson-Phillips Series). *For any $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t, s \in \mathbb{R}$, $t \geq s$,*

$$\begin{aligned} \tau_{t,s}^{(\omega,\lambda,\mathbf{A})} &= \tau_{t-s}^{(\omega,\lambda)} \\ &\quad + \sum_{k\in\mathbb{N}} \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k \tau_{s_k-s}^{(\omega,\lambda)} L_{s_k}^{\mathbf{A}} \tau_{s_{k-1}-s_k}^{(\omega,\lambda)} \cdots \tau_{s_1-s_2}^{(\omega,\lambda)} L_{s_1}^{\mathbf{A}} \tau_{t-s_1}^{(\omega,\lambda)}. \end{aligned}$$

PROOF. Let $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$ and define

$$\begin{aligned} \check{\tau}_{t,s}^{(\omega,\lambda,\mathbf{A})} := \tau_{t-s}^{(\omega,\lambda)} + \sum_{k\in\mathbb{N}} \int_s^t \mathrm{d}s_1 \cdots \\ \times \int_s^{s_{k-1}} \mathrm{d}s_k \tau_{s_k-s}^{(\omega,\lambda)} L_{s_k}^{\mathbf{A}} \tau_{s_{k-1}-s_k}^{(\omega,\lambda)} \cdots \tau_{s_1-s_2}^{(\omega,\lambda)} L_{s_1}^{\mathbf{A}} \tau_{t-s_1}^{(\omega,\lambda)} \end{aligned} \tag{5.8}$$

for any $t \geq s$. This series is absolutely convergent. Indeed, $\tau^{(\omega,\lambda)} := \{\tau_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ is a norm-continuous one-parameter group of contractions, i.e.,

$$\left\| \tau_t^{(\omega,\lambda)} \right\|_{\mathrm{op}} \leq 1, \quad t \in \mathbb{R},$$

whereas, for any $\mathbf{A} \in \mathbf{C}_0^\infty$, the map (5.7) is continuously differentiable and there is a constant $D \in \mathbb{R}^+$ such that

$$\sup_{t\in\mathbb{R}} \| L_t^{\mathbf{A}} \|_{\mathrm{op}} < D, \tag{5.9}$$

because $W_t^{\mathbf{A}} = 0$ for any $t \notin [t_0, t_1]$; i.e., there is no electromagnetic field for times $t \notin [t_0, t_1]$. Here, the notation $\| \cdot \|_{\mathrm{op}}$ stands for the operator norm. By (5.8)–(5.9), it follows that

$$\| \check{\tau}_{t,s}^{(\omega,\lambda,\mathbf{A})} \|_{\mathrm{op}} \leq \mathrm{e}^{D(t-s)}, \quad t, s \in \mathbb{R},\ t \geq s.$$

Now, straightforward computations using (5.4) and (5.7) show that the following "pull through" formula holds:

$$L_t^{\mathbf{A}}(a(\psi)) = a\big(i\mathbf{w}_t^{\mathbf{A}}\psi\big), \quad t \in \mathbb{R},\ \psi \in \ell^2(\mathfrak{L}). \tag{5.10}$$

We therefore infer from (2.6), (5.5), and (5.8) that

$$\check{\tau}_{t,s}^{(\omega,\lambda,\mathbf{A})}(a(\psi)) = a\big(\big(\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}\big)^*(\psi)\big), \quad t \geq s,\ \psi \in \ell^2(\mathfrak{L}), \tag{5.11}$$

for all $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\mathbf{A} \in \mathbf{C}_0^\infty$. Direct computations show, for all $t \geq s$, that $\check{\tau}_{t,s}^{(\omega,\lambda,\mathbf{A})}$ is an automorphism of $\mathcal{U}$: Use the fact that, for all $t \in \mathbb{R}$, $\tau_t^{(\omega,\lambda)}$ is an automorphisms of $\mathcal{U}$ and $L_t^{\mathbf{A}}$ is a bounded symmetric derivation on $\mathcal{U}$, i.e., $L_t^{\mathbf{A}}(B_1^*) = L_t^{\mathbf{A}}(B_1)^*$ and

$$L_t^{\mathbf{A}}(B_1B_2) = L_t^{\mathbf{A}}(B_1)B_2 + B_1L_t^{\mathbf{A}}(B_2) \in \mathcal{U}, \quad B_1, B_2 \in \mathcal{U}.$$

By [7, theorem 5.2.5], the condition (5.11) uniquely defines automorphisms of $\mathcal{U}$. As a consequence, one gets $\check{\tau}_{t,s}^{(\omega,\lambda,\mathbf{A})} = \tau_{t,s}^{(\omega,\lambda,\mathbf{A})}$; see (2.11). □

A straightforward consequence of Theorem 5.1 is that, for any $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, the family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ satisfies (2.12) with

$$\delta_t^{(\omega,\lambda,\mathbf{A})} := \delta^{(\omega,\lambda)} + i[W_t^{\mathbf{A}}, \cdot\,], \quad t \in \mathbb{R}. \tag{5.12}$$

Here, the symmetric derivation $\delta^{(\omega,\lambda)}$ is the (unbounded) generator of the one-parameter group $\tau^{(\omega,\lambda)}$. Indeed, one obtains the following:

COROLLARY 5.2 (Abstract Cauchy Initial Value Problem for $\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}$). *For any* $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, *and* $\mathbf{A} \in \mathbf{C}_0^\infty$, $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ *satisfies* (2.12)*, that is,*

$$\forall t, s \in \mathbb{R},\ t \geq s: \quad \partial_t \tau_{t,s}^{(\omega,\lambda,\mathbf{A})} = \tau_{t,s}^{(\omega,\lambda,\mathbf{A})} \circ \delta_t^{(\omega,\lambda,\mathbf{A})}, \quad \tau_{s,s}^{(\omega,\lambda,\mathbf{A})} := \mathbf{1},$$

on the dense subspace $\mathrm{Dom}(\delta^{(\omega,\lambda)}) \subset \mathcal{U}$.

PROOF. By Theorem 5.1, the family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ obeys the integral equation

$$\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}(B) = \tau_{t-s}^{(\omega,\lambda)}(B) + \int_s^t \tau_{s_1,s}^{(\omega,\lambda)} L_{s_1}^{\mathbf{A}} \tau_{t-s_1}^{(\omega,\lambda)}(B)\mathrm{d}s_1, \quad B \in \mathcal{U},$$

which directly yields the assertion because $\mathbf{A} \in \mathbf{C}_0^\infty$. □

Recall the notation

$$W_{t,s}^{\mathbf{A}} \equiv W_{t,s}^{(\omega,\lambda,\mathbf{A})} := \tau_t^{(\omega,\lambda)}(W_s^{\mathbf{A}}) \in \mathcal{U}, \quad \omega \in \Omega,\ \lambda \in \mathbb{R}_0^+,\ \mathbf{A} \in \mathbf{C}_0^\infty,\ t,s \in \mathbb{R}, \tag{5.13}$$

and the inductive definition (3.14)–(3.15) of multicommutators:

$$[B_1, B_2]^{(2)} := [B_1, B_2] := B_1 B_2 - B_2 B_1, \quad B_1, B_2 \in \mathcal{U}, \tag{5.14}$$

and, for all integers $k > 2$,

$$[B_1, B_2, \ldots, B_{k+1}]^{(k+1)} := [B_1, [B_2, \ldots, B_{k+1}]^{(k)}], \quad B_1, \ldots, B_{k+1} \in \mathcal{U}. \tag{5.15}$$

Then, using (5.2) we rewrite the Dyson-Phillips series of Theorem 5.1 as

$$\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}(B) - \tau_{t-s}^{(\omega,\lambda)}(B) = \sum_{k \in \mathbb{N}} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k [W_{s_k - s, s_k}^{\mathbf{A}}, \ldots, W_{s_1 - s, s_1}^{\mathbf{A}}, \tau_{t-s}^{(\omega,\lambda)}(B)]^{(k+1)} \tag{5.16}$$

for any $B \in \mathcal{U}$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \geq s$.

5.3 Interaction Picture of Dynamics

In contrast to the two-parameter family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t \geq s}$,

$$\left\{\tau_{t_0}^{(\omega,\lambda)} \circ \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A})} \circ \tau_{-t}^{(\omega,\lambda)}\right\}_{t \geq t_0}$$

is a family of *inner* automorphisms of the CAR algebra $\mathcal{U}$; i.e., it can be implemented by conjugation with unitary elements $\mathfrak{V}_{t,t_0}$ of $\mathcal{U}$, similar to Remark 2.1:

$$\tau_{t_0}^{(\omega,\lambda)} \circ \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A})} \circ \tau_{-t}^{(\omega,\lambda)}(B) = \mathfrak{V}_{t,t_0} B \mathfrak{V}_{t,t_0}^*, \quad B \in \mathcal{U}.$$

On the other hand, by using two times the stationarity of the KMS state $\varrho^{(\beta,\omega,\lambda)}$ with respect to the unperturbed dynamics (cf. (5.3)) as well as (5.2), we observe that the time evolution (2.13) of the state of the fermion system equals

$$\begin{aligned}\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(B) &= \varrho^{(\beta,\omega,\lambda)} \circ \tau_{t_0}^{(\omega,\lambda)} \circ \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A})}(B) \\ &= \varrho^{(\beta,\omega,\lambda)}\big(\mathfrak{V}_{t,t_0} \tau_t^{(\omega,\lambda,\mathbf{A})}(B) \mathfrak{V}_{t,t_0}^*\big) = \varrho^{(\beta,\omega,\lambda)}(\mathfrak{U}_t^* B \mathfrak{U}_t\big)\end{aligned}$$

for any $t \geq t_0$, where

$$\mathfrak{U}_t := \tau_{-t}^{(\omega,\lambda)}\big(\mathfrak{V}_{t,t_0}^*\big), \quad t \geq t_0. \tag{5.17}$$

This family of unitary elements of $\mathcal{U}$ turns out to be within the domain $\mathrm{Dom}(\delta^{(\omega,\lambda)})$ of the (unbounded) generator $\delta^{(\omega,\lambda)}$ of the one-parameter group $\tau^{(\omega,\lambda)}$ of automorphisms. These properties are quite useful to show in Section 5.4 both the existence of the energy increment (3.8) as well as Theorem 3.2.

The above heuristics is proven in the following theorem:

THEOREM 5.3 (Interaction Picture of Dynamics). *For any $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$, there is a family*

$$\{\mathfrak{U}_t \equiv \mathfrak{U}_t^{(\omega,\lambda,\mathbf{A})}\}_{t \geq t_0} \subset \mathrm{Dom}(\delta^{(\omega,\lambda)})$$

of unitary elements of $\mathcal{U}$ such that, for all $\beta \in \mathbb{R}^+$, $t \geq t_0$, and $B \in \mathcal{U}$,

$$\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(B) = \varrho^{(\beta,\omega,\lambda)}(\mathfrak{U}_t^* B \mathfrak{U}_t).$$

PROOF. The arguments to prove this theorem are relatively standard for autonomous perturbations of KMS states; see [7, sec. 5.4.1.]. We adapt them to the nonautonomous case as suggested in [7, sec. 5.4.4., prop. 5.4.26.]. However, in contrast to [7, secs. 5.4.1., 5.4.4.], the situation we treat here requires more care because the symmetric derivation $\delta^{(\omega,\lambda)}$ is *unbounded*.

For any $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\mathbf{A} \in \mathbf{C}_0^\infty$, we define the family $\{\mathfrak{U}_{t,s}\}_{t,s\in\mathbb{R}} \subset \mathcal{U}$ by the series

$$\mathfrak{V}_{t,s} \equiv \mathfrak{V}_{t,s}^{(\omega,\lambda,\mathbf{A})} := \mathbf{1} + \sum_{k\in\mathbb{N}} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k W_{s_k,s_k}^{\mathbf{A}} \cdots W_{s_1,s_1}^{\mathbf{A}}, \tag{5.18}$$

where we recall that $W_{t,s}^{\mathbf{A}} \equiv W_{t,s}^{(\omega,\lambda,\mathbf{A})} \in \mathcal{U}$ is defined by (5.13) for any $t, s \in \mathbb{R}$. The series is well-defined in the Banach space

$$\mathcal{Y} := (\mathrm{Dom}(\delta^{(\omega,\lambda)}), \|\cdot\|_{\delta^{(\omega,\lambda)}}), \tag{5.19}$$

where $\|\cdot\|_{\delta^{(\omega,\lambda)}}$ stands for the graph norm of the closed operator $\delta^{(\omega,\lambda)}$. In particular,

$$\{\mathfrak{V}_{t,s}\}_{t,s\in\mathbb{R}} \subset \mathrm{Dom}(\delta^{(\omega,\lambda)}). \tag{5.20}$$

Indeed, the strongly continuous group $\tau^{(\omega,\lambda)}$ on $\mathcal{U}$ defines, by restriction, a strongly continuous group on $\mathcal{Y}$. For more details, see, e.g., [14, sec. II.5.a, 5.2 Proposition]. Observe also from the strong continuity and group property in $\mathcal{Y}$ of the restriction of $\tau^{(\omega,\lambda)}$ to the space $\mathrm{Dom}(\delta^{(\omega,\lambda)})$ that

$$\big\|\tau_t^{(\omega,\lambda)}|_{\mathrm{Dom}(\delta^{(\omega,\lambda)})}\big\|_{\mathcal{B}(\mathcal{Y})} \leq D_1 e^{D_2|t|} \tag{5.21}$$

for some finite constants $D_1, D_2 \in \mathbb{R}^+$ and all $t \in \mathbb{R}$. Here, $\mathcal{B}(\mathcal{Y})$ is the Banach space of bounded operators acting on $\mathcal{Y}$. Moreover, for any $\mathbf{A} \in \mathbf{C}_0^\infty$, $s \mapsto W_s^{\mathbf{A}}$ is a smooth, compactly supported map from $\mathbb{R}$ to $\mathcal{Y}$. Since $\delta^{(\omega,\lambda)}$ is a symmetric derivation, it follows that the series (5.18) absolutely converges in the Banach space $\mathcal{Y}$ and

$$\mathfrak{V}_{t,s} = \mathbf{1} + \sum_{k\in\mathbb{N}} i^k \int_s^t \mathrm{d}s_k \cdots \int_{s_2}^t \mathrm{d}s_1 W_{s_k,s_k}^{\mathbf{A}} \cdots W_{s_1,s_1}^{\mathbf{A}},$$

where the right-hand side of this equation also absolutely converges in $\mathcal{Y}$. Therefore, for any $t, s \in \mathbb{R}$, the operator $\mathfrak{V}_{t,s}$ obeys the integral equation

$$\mathfrak{V}_{t,s} = \mathbf{1} + i \int_s^t \mathfrak{V}_{s_1,s} W^{\mathbf{A}}_{s_1,s_1} \, \mathrm{d}s_1 = \mathbf{1} + i \int_s^t W^{\mathbf{A}}_{s_1,s_1} \mathfrak{V}_{t,s_1} \, \mathrm{d}s_1 \tag{5.22}$$

in $\mathcal{Y}$. The families $\{\mathfrak{U}_{t,s}\}_{t,s\in\mathbb{R}}$ and $\{W^{\mathbf{A}}_{t,t}\}_{t\in\mathbb{R}}$ are both continuous in $\mathcal{Y}$ and $\delta^{(\omega,\lambda)}$ is a symmetric derivation. As a consequence, (5.22) implies that, for any $t, s \in \mathbb{R}$,

$$\partial_t \mathfrak{V}_{t,s} = i \mathfrak{V}_{t,s} W^{\mathbf{A}}_{t,t} \quad \text{and} \quad \partial_s \mathfrak{V}_{t,s} = -i W^{\mathbf{A}}_{s,s} \mathfrak{V}_{t,s}, \tag{5.23}$$

both in the Banach space $\mathcal{Y}$ and thus in $\mathcal{U}$. Since $W^{\mathbf{A}}_{t,t} = (W^{\mathbf{A}}_{t,t})^*$, by using the norm-continuity of the map $B \mapsto B^*$ on $\mathcal{U}$, we compute from (5.23) that

$$\mathbf{1} - \mathfrak{V}^*_{t,s}\mathfrak{V}_{t,s} = \int_s^t \partial_{s_1} \{\mathfrak{V}^*_{t,s_1}\mathfrak{V}_{t,s_1}\} \mathrm{d}s_1 = 0.$$
$$\mathbf{1} - \mathfrak{V}_{t,s}\mathfrak{V}^*_{t,s} = \int_t^s \partial_{s_1} \{\mathfrak{V}_{s_1,s}\mathfrak{V}^*_{s_1,s}\} \mathrm{d}s_1 = 0.$$

In other words, $\{\mathfrak{V}_{t,s}\}_{t,s\in\mathbb{R}}$ is a family of unitary elements of $\mathrm{Dom}(\delta^{(\omega,\lambda)}) \subset \mathcal{U}$, by (5.20).

Now we define the family $\{\mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t}\}_{s,t\in\mathbb{R}}$ of bounded operators acting on the Banach space $\mathcal{U}$ by

$$\mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t}(B) := \tau^{(\omega,\lambda)}_{-s}\big(\mathfrak{V}_{t,s}\tau^{(\omega,\lambda)}_t(B)\mathfrak{V}^*_{t,s}\big), \quad B \in \mathcal{U}. \tag{5.24}$$

Clearly, for any $B \in \mathcal{U}$, the map

$$(s,t) \mapsto \mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t}(B) \in \mathcal{U}$$

from $\mathbb{R}^2$ to $\mathcal{U}$ is continuous. Moreover, by construction, $\mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{t,t} = \mathbf{1}$ and for all $B \in \mathrm{Dom}(\delta^{(\omega,\lambda)})$ and $s, t \in \mathbb{R}$,

$$\mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t}(B) \in \mathrm{Dom}(\delta^{(\omega,\lambda)}) = \mathrm{Dom}\big(\delta^{(\omega,\lambda,\mathbf{A})}_s\big),$$

because $\tau^{(\omega,\lambda)}_t$ preserves the (dense) subspace $\mathrm{Dom}(\delta^{(\omega,\lambda)}) \subset \mathcal{U}$. Therefore, we infer from (5.2), (5.12), and (5.23) that

$$\forall s,t \in \mathbb{R}: \quad \partial_s \mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t} = -\delta^{(\omega,\lambda,\mathbf{A})}_s \circ \mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t}, \quad \mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{t,t} = \mathbf{1}, \tag{5.25}$$

whereas

$$\forall s,t \in \mathbb{R}: \quad \partial_t \mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t} = \mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t} \circ \delta^{(\omega,\lambda,\mathbf{A})}_t, \quad \mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,s} = \mathbf{1}, \tag{5.26}$$

both in the strong sense in $\mathrm{Dom}(\delta^{(\omega,\lambda)}) \subset \mathcal{U}$. In particular, by Corollary 5.2, the families $\{\tau^{(\omega,\lambda,\mathbf{A})}_{t,s}\}_{t\geq s}$ and $\{\mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t}\}_{s,t\in\mathbb{R}}$ satisfy the equality

$$\tau^{(\omega,\lambda,\mathbf{A})}_{t,s}(B) - \mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s,t}(B) = \int_s^t \partial_{s_1}\{\tau^{(\omega,\lambda,\mathbf{A})}_{s_1,s}\mathfrak{W}^{(\omega,\lambda,\mathbf{A})}_{s_1,t}(B)\}\mathrm{d}s_1 = 0 \tag{5.27}$$

for any $B \in \mathrm{Dom}(\delta^{(\omega,\lambda)})$ and $t \geq s$. Note that we use the strong continuity of the family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ with respect to $t \in \mathbb{R}$ to show from Corollary 5.2 and (5.25) that

$$\partial_{s_1}\{\tau_{s_1,s}^{(\omega,\lambda,\mathbf{A})}\mathfrak{W}_{s_1,t}^{(\omega,\lambda,\mathbf{A})}(B)\} = 0$$

for any $B \in \mathrm{Dom}(\delta^{(\omega,\lambda)})$ and $t \geq s$. The domain $\mathrm{Dom}(\delta^{(\omega,\lambda)})$ is dense in $\mathcal{U}$, and both operators $\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}$ and $\mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})}$ are bounded. As a consequence, (5.27) yields

$$\tau_{t,s}^{(\omega,\lambda,\mathbf{A})} = \mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})} \tag{5.28}$$

for any $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \geq s$.

Now we use Equation (5.17) to define the family $\{\mathfrak{U}_t\}_{t\geq t_0}$. Since, for any $t \in \mathbb{R}$, $\tau_{-t}^{(\omega,\lambda)}$ is an automorphism of $\mathcal{U}$ that preserves the domain $\mathrm{Dom}(\delta^{(\omega,\lambda)})$, we deduce from (5.20) and the unitarity of $\mathfrak{V}_{t,s}$ that

$$\{\mathfrak{U}_t\}_{t\geq t_0} \subset \mathrm{Dom}(\delta^{(\omega,\lambda)})$$

is a family of unitary elements of $\mathcal{U}$. Note indeed that $\mathrm{Dom}(\delta^{(\omega,\lambda)})$ is a $*$-algebra, since $\delta^{(\omega,\lambda)}$ is a symmetric derivation. Moreover, from (2.13), (5.17), (5.24), and (5.28) combined with the stationarity of the KMS state $\varrho^{(\beta,\omega,\lambda)}$ with respect to the unperturbed dynamics (cf. (5.3)), we arrive at the assertion, as explained in Equation (5.17). □

The proof of Theorem 5.3 gives supplementary information on the dynamics. This is not used in the present paper, but it can be employed to uniquely define dynamics for systems of interacting fermions on the lattice, as discussed at the end of Section 2.4.

First, by (5.24), $\{\mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})}\}_{s,t\in\mathbb{R}}$ is a family of bounded operators acting on the Banach space $\mathcal{U}$ that of course extends $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ to all $s,t \in \mathbb{R}$; see (5.28). Moreover, it is the unique *fundamental solution* of a nonautonomous evolution equation. By fundamental solution, we mean here that the family $\{\mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})}\}_{s,t\in\mathbb{R}}$ of bounded operators acting on $\mathcal{U}$ is strongly continuous, conserves the domain

$$\mathrm{Dom}(\delta_t^{(\omega,\lambda,\mathbf{A})}) = \mathrm{Dom}(\delta^{(\omega,\lambda)}),$$

satisfies

$$\mathfrak{W}_{\cdot,t}^{(\omega,\lambda,\mathbf{A})}(B) \in C^1(\mathbb{R}; (\mathrm{Dom}(\delta^{(\omega,\lambda)}), \|\cdot\|)),$$
$$\mathfrak{W}_{s,\cdot}^{(\omega,\lambda,\mathbf{A})}(B) \in C^1(\mathbb{R}; (\mathrm{Dom}(\delta^{(\omega,\lambda)}), \|\cdot\|)),$$

for all $B \in \mathrm{Dom}(\delta^{(\omega,\lambda)})$, and solves the abstract Cauchy initial value problem (5.25) on $\mathrm{Dom}(\delta^{(\omega,\lambda)})$.

PROPOSITION 5.4 (Evolution Equations for $\mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})}$). *For $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\mathbf{A} \in \mathbf{C}_0^\infty$, $\{\mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})}\}_{s,t\in\mathbb{R}}$ has the following properties*:

(i) *It satisfies the Chapman-Kolmogorov property,*

$$\forall t, r, s \in \mathbb{R}: \quad \mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})} = \mathfrak{W}_{s,r}^{(\omega,\lambda,\mathbf{A})} \mathfrak{W}_{r,t}^{(\omega,\lambda,\mathbf{A})}.$$

(ii) *It is the unique fundamental solution of the Cauchy initial value problem*

$$\forall s, t \in \mathbb{R}: \quad \partial_s \mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})} = -\delta_s^{(\omega,\lambda,\mathbf{A})} \circ \mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})}, \quad \mathfrak{W}_{t,t}^{(\omega,\lambda,\mathbf{A})} = \mathbf{1}.$$

(iii) *It solves on* $\mathrm{Dom}(\delta^{(\omega,\lambda)})$ *the abstract Cauchy initial value problem*

$$\forall s, t \in \mathbb{R}: \quad \partial_t \mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})} = \mathfrak{W}_{s,t}^{(\omega,\lambda,\mathbf{A})} \circ \delta_t^{(\omega,\lambda,\mathbf{A})}, \quad \mathfrak{W}_{s,s}^{(\omega,\lambda,\mathbf{A})} = \mathbf{1}.$$

PROOF. Use (5.24)–(5.26) and an argument similar to (5.27). We omit the details. □

5.4 Internal Energy Increment and Heat Production

Recall that the internal energy increment is defined by (3.8), that is,

$$\mathbf{S}^{(\omega,\mathbf{A})}(t) \equiv \mathbf{S}^{(\beta,\omega,\lambda,\mathbf{A})}(t) := \lim_{L\to\infty} \left\{ \rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\left(H_L^{(\omega,\lambda)}\right) - \varrho^{(\beta,\omega,\lambda)}\left(H_L^{(\omega,\lambda)}\right) \right\} \tag{5.29}$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \in \mathbb{R}$. To show that it is well-defined and has finite value for all times, we use the interaction picture of the dynamics described in Theorem 5.3:

THEOREM 5.5 (Existence of the Internal Energy Increment). *For any* $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, *and* $t \geq t_0$,

$$\mathbf{S}^{(\omega,\mathbf{A})}(t) = -i\varrho^{(\beta,\omega,\lambda)}\left(\mathfrak{U}_t^* \delta^{(\omega,\lambda)}(\mathfrak{U}_t)\right) \in \mathbb{R}$$

with $\{\mathfrak{U}_t\}_{t\geq t_0} \subset \mathrm{Dom}(\delta^{(\omega,\lambda)})$ *being defined in Theorem* 5.3.

PROOF. $\mathfrak{U}_t \in \mathrm{Dom}(\delta^{(\omega,\lambda)})$ and, by explicit computations that use equations (5.17)–(5.18) together with the "pull-through" formula (5.10),

$$\delta^{(\omega,\lambda)}(\mathfrak{U}_t) = \lim_{L\to\infty} \left\{ i\left[H_L^{(\omega,\lambda)}, \mathfrak{U}_t\right] \right\} \in \mathcal{U},$$

whereas one obviously has

$$\varrho^{(\beta,\omega,\lambda)}\left(\mathfrak{U}_t^*\left[H_L^{(\omega,\lambda)}, \mathfrak{U}_t\right]\right) = \rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\left(H_L^{(\omega,\lambda)}\right) - \varrho^{(\beta,\omega,\lambda)}\left(H_L^{(\omega,\lambda)}\right),$$

by Theorem 5.3. We obtain the assertion by combining (5.29) with these two equalities and the continuity of states. □

Therefore, $\mathbf{S}^{(\omega,\mathbf{A})}$ is a map from $\mathbb{R}$ to $\mathbb{R}$. Now, by the Pusz-Woronowicz theorem (see, e.g., [7, theorem 5.3.22]), it is well-known that (τ, β)-KMS states ϱ are *passive states*, that is,

$$-i\varrho(U^*\delta(U)) \geq 0$$

for all unitaries U both in the domain of definition of the generator δ of the group τ and in the connected component of the identity of the group of all unitary elements of the CAR algebra with the norm topology. The latter, together with equations (5.17) and (5.18), directly implies the positivity of the internal energy increment $\mathbf{S}^{(\omega,\mathbf{A})}$:

COROLLARY 5.6 (Positivity of the Internal Energy Increment). *For any* $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, *and all* $t \geq t_0$, $\mathbf{S}^{(\omega,\mathbf{A})}(t) \geq 0$.

Moreover, for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \geq t_0$, we also infer from [17, theorem 1.1] and Theorem 5.3 that

$$-i\varrho^{(\beta,\omega,\lambda)}\big(\mathfrak{U}_t^*\delta^{(\omega,\lambda)}(\mathfrak{U}_t)\big) = \beta^{-1}\mathrm{S}\big(\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}\big|\varrho^{(\beta,\omega,\lambda)}\big)$$

with S being the relative entropy defined by (3.4). See also (A.9) and recall that $\mathrm{S} = S_{\mathcal{U}}$. By Definition 3.1, we thus recover the heat production $\mathbf{Q}^{(\omega,\mathbf{A})}$ from Theorem 5.5:

COROLLARY 5.7 (Heat Production as an Internal Energy Increment).
For any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, *and* $\mathbf{A} \in \mathbf{C}_0^\infty$, $\mathbf{S}^{(\omega,\mathbf{A})} = \mathbf{Q}^{(\omega,\mathbf{A})}$.

Finally, Theorems 5.3 and 5.5 also yield a simple and convenient expression of the *total* energy increment (3.7)–(3.9) delivered to the system by the electromagnetic field at time $t \in \mathbb{R}$:

THEOREM 5.8 (Total Energy Increment and Electromagnetic Work). *For any* $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, *and* $t \geq t_0$,

$$\mathbf{S}^{(\omega,\mathbf{A})}(t) + \mathbf{P}^{(\omega,\mathbf{A})}(t) = \int_{t_0}^{t} \rho_s^{(\beta,\omega,\lambda,\mathbf{A})}\big(\partial_s W_s^{\mathbf{A}}\big)\mathrm{d}s.$$

PROOF. The proof is an extension of the one of [7, lemma 5.4.27] to the *unbounded* symmetric derivation $\delta^{(\omega,\lambda)}$.

By (5.17) and the stationarity of the KMS state $\varrho^{(\beta,\omega,\lambda)}$ with respect to the unperturbed dynamics (cf. (5.3)), we first observe that, for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \geq t_0$,

$$\varrho^{(\beta,\omega,\lambda)}\big(\mathfrak{U}_t^*\delta^{(\omega,\lambda)}(\mathfrak{U}_t)\big) = \varrho^{(\beta,\omega,\lambda)}\big(\mathfrak{V}_{t,t_0}\delta^{(\omega,\lambda)}(\mathfrak{V}_{t,t_0}^*)\big) \tag{5.30}$$

with the unitary elements $\mathfrak{V}_{t,t_0}$ being defined by (5.18).

The maps

$$t \mapsto \mathfrak{V}_{t,t_0} \quad \text{and} \quad t \mapsto \delta^{(\omega,\lambda)}\big(\mathfrak{V}_{t,t_0}^*\big)$$

from $\mathbb{R}$ to $\mathrm{Dom}(\delta^{(\omega,\lambda)})$ are continuously differentiable in the Banach spaces $\mathcal{Y}$ and $\mathcal{U}$, respectively. See (5.19) and (5.23). Therefore, the map

$$t \mapsto \partial_t\big\{\varrho^{(\beta,\omega,\lambda)}\big(\mathfrak{V}_{t,t_0}\delta^{(\omega,\lambda)}\big(\mathfrak{V}_{t,t_0}^*\big)\big)\big\}$$

from $\mathbb{R}$ to $\mathbb{R}$ is also continuously differentiable and, from (5.23) and the fact that $\delta^{(\omega,\lambda)}$ is a symmetric derivation, we compute that, for all $t \in \mathbb{R}$,

$$\partial_t\big\{\varrho^{(\beta,\omega,\lambda)}\big(\mathfrak{V}_{t,t_0}\delta^{(\omega,\lambda)}\big(\mathfrak{V}_{t,t_0}^*\big)\big)\big\} = - i\varrho^{(\beta,\omega,\lambda)}\big(\mathfrak{V}_{t,t_0}\big\{\delta^{(\omega,\lambda)}\big(W_{t,t}^{\mathbf{A}}\big)\big\}\mathfrak{V}_{t,t_0}^*\big). \tag{5.31}$$

On the other hand, using again (5.23) we observe that

$$\partial_t\{\mathfrak{V}_{t,t_0} W_{t,t}^{\mathbf{A}} \mathfrak{V}_{t,t_0}^*\} = \mathfrak{V}_{t,t_0}(\partial_t W_{t,t}^{\mathbf{A}})\mathfrak{V}_{t,t_0}^*$$

for any $t \in \mathbb{R}$, which, combined with the identity

$$\delta^{(\omega,\lambda)}(W_{t,t}^{\mathbf{A}}) = \partial_t W_{t,t}^{\mathbf{A}} - \tau_t^{(\omega,\lambda)}(\partial_t W_t^{\mathbf{A}})$$

yields

$$\mathfrak{V}_{t,t_0}\{\delta^{(\omega,\lambda)}(W_{t,t}^{\mathbf{A}})\}\mathfrak{V}_{t,t_0}^* = \partial_t\{\mathfrak{V}_{t,t_0} W_{t,t}^{\mathbf{A}} \mathfrak{V}_{t,t_0}^*\} - \mathfrak{V}_{t,t_0}\tau_t^{(\omega,\lambda)}(\partial_t W_t^{\mathbf{A}})\mathfrak{V}_{t,t_0}^*.$$

Using this equality together with (5.31), we thus find that, for any $t \in \mathbb{R}$,

$$(5.32)\quad \partial_t\{\varrho^{(\beta,\omega,\lambda)}(\mathfrak{V}_{t,t_0}\delta^{(\omega,\lambda)}(\mathfrak{V}_{t,t_0}^*))\} = \\ -i\varrho^{(\beta,\omega,\lambda)}(\partial_t\{\mathfrak{V}_{t,t_0} W_{t,t}^{\mathbf{A}} \mathfrak{V}_{t,t_0}^*\}) + i\varrho^{(\beta,\omega,\lambda)}(\mathfrak{V}_{t,t_0}\tau_t^{(\omega,\lambda)}(\partial_t W_t^{\mathbf{A}})\mathfrak{V}_{t,t_0}^*).$$

Now, for $t \in \mathbb{R}$, we use Equations (2.13), (5.3), (5.24), (5.28), (5.30), and (5.32) to arrive at

$$\partial_t\{\varrho^{(\beta,\omega,\lambda)}(\mathfrak{U}_t^*\delta^{(\omega,\lambda)}(\mathfrak{U}_t))\} = \\ -i\varrho^{(\beta,\omega,\lambda)}(\partial_t\{\mathfrak{V}_{t,t_0} W_{t,t}^{\mathbf{A}} \mathfrak{V}_{t,t_0}^*\}) + i\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(\partial_t W_t^{\mathbf{A}}).$$

We next integrate this last equality by using $\mathfrak{V}_{t_0,t_0} = \mathfrak{U}_{t_0} = \mathbf{1}$ to get

$$(5.33)\quad \varrho^{(\beta,\omega,\lambda)}(\mathfrak{U}_t^*\delta^{(\omega,\lambda)}(\mathfrak{U}_t)) = \\ i\int_{t_0}^t \rho_s^{(\beta,\omega,\lambda,\mathbf{A})}(\partial_s W_s^{\mathbf{A}})\mathrm{d}s - i\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(W_t^{\mathbf{A}}) + i\varrho^{(\beta,\omega,\lambda)}(W_{t_0}^{\mathbf{A}})$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \geq t_0$. The assertion then follows from (5.33) combined with (3.9) and Theorem 5.5. □

Following the terminology of [7, sec. 5.4.4.] with their definition of L^P, Theorem 5.8 means that the total energy increment (3.7) is equal to the *work* performed on the system by the electromagnetic field at time $t \geq t_0$. Moreover, Theorem 5.8 leads to the real analyticity of the internal energy increment with respect to the field strength $\eta \in \mathbb{R}$:

COROLLARY 5.9 (Real Analyticity of the Internal Energy Increment). *For any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \geq t_0$, $\mathbf{S}^{(\omega,\eta\mathbf{A})}(t)$ is a real analytic function of $\eta \in \mathbb{R}$.*

PROOF. Use Theorem 5.8 and write the terms $\mathbf{P}^{(\omega,\eta\mathbf{A})}(t)$ and

$$\int_{t_0}^t \rho_s^{(\beta,\omega,\lambda,\eta\mathbf{A})}(\partial_s W_s^{\eta\mathbf{A}})\mathrm{d}s$$

as Dyson-Phillips series in terms of multicommutators; see (2.13) and (5.16). Observe finally that both maps

$$\eta \mapsto W_s^{\eta \mathbf{A}} \in \mathcal{U} \quad \text{and} \quad \eta \mapsto \partial_s W_s^{\eta \mathbf{A}} \in \mathcal{U}$$

are real analytic with infinite analyticity radius. □

5.5 Behavior of the Internal Energy Increment at Small Fields

We study here the asymptotic behavior of $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)} \equiv \mathbf{S}^{(\beta,\omega,\lambda,\eta\mathbf{A}_l)}$ at small field strength $\eta \in \mathbb{R}$ and large space scale $l \in \mathbb{R}^+$. In fact, in view of Corollary 5.7 saying that $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)} = \mathbf{Q}^{(\omega,\eta\mathbf{A}_l)}$, we prove here Theorem 3.4. Recall that $\mathbf{A}_l \in \mathbf{C}_0^\infty$ is defined by (3.13), that is,

$$\mathbf{A}_l(t,x) := \mathbf{A}(t,l^{-1}x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^d, \tag{5.34}$$

for any $\mathbf{A} \in \mathbf{C}_0^\infty$ and $l \in \mathbb{R}^+$.

Using equations (2.5), (2.13), (3.5), (5.3), and (5.16) we first observe that

$$\begin{aligned} &\rho_t^{(\beta,\omega,\lambda,\eta\mathbf{A}_l)}\big(H_L^{(\omega,\lambda)}\big) - \rho_{t_0}^{(\beta,\omega,\lambda,\eta\mathbf{A}_l)}\big(H_L^{(\omega,\lambda)}\big) \\ &\quad = \sum_{x\in\Lambda_L}\sum_{z\in\mathfrak{L},|z|\le 1} \langle \mathfrak{e}_x, (\Delta_{\mathrm{d}} + \lambda V_\omega)\mathfrak{e}_{x+z}\rangle \mathbf{1}[x+z\in\Lambda_L] \sum_{k\in\mathbb{N}} i^k \\ &\qquad \times \int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2 \cdots \int_{t_0}^{s_{k-1}} \mathrm{d}s_k \\ &\qquad \varrho^{(\beta,\omega,\lambda)}\big(\big[W_{s_k-t_0,s_k}^{\eta\mathbf{A}_l}, \ldots, W_{s_1-t_0,s_1}^{\eta\mathbf{A}_l}, \tau_{t-t_0}^{(\omega,\lambda)}(a_x^* a_{x+z})\big]^{(k+1)}\big) \end{aligned} \tag{5.35}$$

for any $L, l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\eta \in \mathbb{R}$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \geq t_0$. Recall that the time-dependent electromagnetic perturbation $W_{t,s}^{\mathbf{A}}$ is defined by (5.13). See also (5.14)–(5.15) for the precise definition of multicommutators.

Therefore, in order to write $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}$ in terms of multicommutators, we prove the following lemma by using tree-decay bounds:

LEMMA 5.10 (Bounds on Multicommutators). *For any $\mathbf{A} \in \mathbf{C}_0^\infty$, there is $\eta_0 \in \mathbb{R}^+$ such that, for any $l, \varepsilon \in \mathbb{R}^+$, there is a ball*

$$B(0,R) := \{x \in \mathfrak{L} : |x| \leq R\} \tag{5.36}$$

of radius $R \in \mathbb{R}^+$ centered at 0 such that, for all $|\eta| \in [0,\eta_0]$, $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $t_0 \leq s_1, \ldots, s_k \leq t$,

$$\begin{aligned} &\sum_{x\in\Lambda_L\setminus B_R}\sum_{z\in\mathfrak{L},|z|\le 1}\sum_{k\in\mathbb{N}} \frac{(t-t_0)^k}{k!} \\ &\qquad \big|\varrho^{(\beta,\omega,\lambda)}\big(\big[W_{s_k-t_0,s_k}^{\eta\mathbf{A}_l}, \ldots, W_{s_1-t_0,s_1}^{\eta\mathbf{A}_l}, \tau_{t-t_0}^{(\omega,\lambda)}(a_x^* a_{x+z})\big]^{(k+1)}\big)\big| \leq \varepsilon. \end{aligned}$$

PROOF. We first need to bound the $(k+1)$-fold multicommutator

$$\big[W_{s_k-t_0,s_k}^{\mathbf{A}}, \ldots, W_{s_1-t_0,s_1}^{\mathbf{A}}, \tau_{t-t_0}^{(\omega,\lambda)}(a_x^* a_{x+z})\big]^{(k+1)}$$

for any $k \in \mathbb{N}$, $x \in \Lambda_L$, and $z \in \mathfrak{L}$ so that $|z| \leq 1$. This is done by using tree-decay bounds as explained in Section 4. Indeed, by (5.34), for any $l \in \mathbb{R}^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$, there exists a finite subset $\widetilde{\Lambda}_l \in \mathcal{P}_f(\mathfrak{L})$ such that $\mathbf{A}_l(t,x) = 0$ for all $x \in \mathfrak{L} \setminus \widetilde{\Lambda}_l$ and $t \in \mathbb{R}$. Then, we infer from (5.6) and (5.13) that, for all $l \in \mathbb{R}^+$, $x, y \in \mathfrak{L}$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t, \eta \in \mathbb{R}$, there are constants $D_{x,y}^{\eta \mathbf{A}_l}(t) \in \mathbb{C}$ such that

$$W_{s_1,s_2}^{\eta \mathbf{A}_l} = \sum_{x \in \widetilde{\Lambda}_l} \sum_{\substack{z \in \mathfrak{L} \\ |z| \leq 1}} D_{x,x+z}^{\eta \mathbf{A}_l}(s_2) \tau_{s_1}^{(\omega,\lambda)}\left(a_x^* a_{x+z}\right) \tag{5.37}$$

for any $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $s_1, s_2 \in \mathbb{R}$. Here, the constants $D_{x,y}^{\eta \mathbf{A}_l}(t)$ are always of order η:

$$\sup_{t \in \mathbb{R},\ x,y \in \mathfrak{L}} \left| D_{x,y}^{\eta \mathbf{A}_l}(t) \right| \leq K_\eta \tag{5.38}$$

with

$$K_\eta := \|\Delta_{\mathrm{d}}\|_{\mathrm{op}} \left| \exp\left\{ i \left| \eta \right| \max_{\substack{(t,x) \in \mathbb{R} \times \mathbb{R}^d \\ z \in \mathfrak{L} \\ |z| \leq 1}} \left| \left[\mathbf{A}(t,x) \right](z) \right| \right\} - 1 \right| = \mathcal{O}(|\eta|). \tag{5.39}$$

(Recall that $\|\cdot\|_{\mathrm{op}}$ is the operator norm.) Therefore, using Corollary 4.3 we deduce that, for every $\epsilon \in \mathbb{R}^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t > t_0$, there is a constant $D \in \mathbb{R}^+$ such that, for any $k \in \mathbb{N}$, $L, l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\eta \in \mathbb{R}$, $s_1, \ldots, s_k \in [t_0, t]$, and $R > R_l$,

$$\begin{aligned} \sum_{x \in \Lambda_L \setminus B_R} \sum_{\substack{z \in \mathfrak{L} \\ |z| \leq 1}} \left| \varrho^{(\beta,\omega,\lambda)}\left(\left[W_{s_k - t_0, s_k}^{\eta \mathbf{A}_l}, \ldots, W_{s_1 - t_0, s_1}^{\eta \mathbf{A}_l}, \tau_{t-t_0}^{(\omega,\lambda)}(a_x^* a_{x+z}) \right]^{(k+1)} \right) \right| \leq \\ |\widetilde{\Lambda}_l| \, |\mathcal{T}_{k+1}| \Bigg[\sum_{\substack{x \in \mathfrak{L} \\ |x| \geq R - R_l}} \frac{K_\eta D}{1 + |x|^{d+\epsilon}} \Bigg] \Bigg[\sum_{x \in \mathfrak{L}} \frac{K_\eta D}{1 + |x|^{d+\epsilon}} \Bigg]^{k-1}, \end{aligned} \tag{5.40}$$

with $B(0, R)$ being the ball (5.36) of radius $R \in \mathbb{R}^+$ centered at 0 and where $|\widetilde{\Lambda}_l|$ is the volume of the finite subset $\widetilde{\Lambda}_l \in \mathcal{P}_f(\mathfrak{L})$ with radius

$$R_l := \max\{|x| : x \in \widetilde{\Lambda}_l\} \in \mathbb{R}^+, \quad l \in \mathbb{R}^+. \tag{5.41}$$

Note that there exists a finite constant $D \in \mathbb{R}^+$ such that $R_l \leq lD$ for all $l \in \mathbb{R}^+$.

From (5.6) and (5.13) it follows that $W_{t,s}^{\mathbf{A}} = 0$ for any $t \geq t_1$, where t_1 is the time when the electromagnetic potential is switched off. Therefore, without loss of generality, we only consider times $t \in (t_0, t_1]$ with $t_1 > t_0$. Thus, take $\eta_0 \in \mathbb{R}^+$ sufficiently small to imply

$$\sum_{x \in \mathfrak{L}} \frac{K_\eta D}{1 + |x|^{d+\epsilon}} \leq \sum_{x \in \mathfrak{L}} \frac{K_{\eta_0} D}{1 + |x|^{d+\epsilon}} \leq \frac{1}{2(t_1 - t_0)}$$

for all $|\eta| \in [0, \eta_0]$. Then, using $|\mathcal{T}_{k+1}| = k!$ and the upper bound (5.40) we arrive at

$$\sum_{x\in\Lambda_L\backslash B_R} \sum_{z\in\mathfrak{L},|z|\leq 1} \left|\varrho^{(\beta,\omega,\lambda)}\left(\left[W^{\eta\mathbf{A}_l}_{s_k-t_0,s_k},\ldots,W^{\eta\mathbf{A}_l}_{s_1-t_0,s_1},\tau^{(\omega,\lambda)}_{t-t_0}(a^*_x a_{x+z})\right]^{(k+1)}\right)\right| \leq \frac{k!}{2^{k-1}\left(t_1-t_0\right)^{k-1}}|\widetilde{\Lambda}_l| \sum_{x\in\mathfrak{L},|x|\geq R-R_l} \frac{K_\eta D}{1+|x|^{d+\epsilon}} \tag{5.42}$$

for all $|\eta| \in [0, \eta_0]$ and any $L, l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}^+_0$, $k \in \mathbb{N}$, $t \in (t_0, t_1]$, and $s_1, \ldots, s_k \in [t_0, t]$. Therefore, we get the assertion from (5.42) by choosing $R \in \mathbb{R}^+$ such that

$$2(t_1-t_0)|\widetilde{\Lambda}_l| \sum_{\substack{x\in\mathfrak{L}\\ |x|\geq R-R_l}} \frac{K_{\eta_0} D}{1+|x|^{d+\epsilon}} \leq \varepsilon$$

for some fixed arbitrarily chosen parameter $\varepsilon \in \mathbb{R}^+$. □

For any $\mathbf{A} \in \mathbf{C}^\infty_0$, this lemma implies the existence of a constant $\eta_0 \in \mathbb{R}^+$ such that, for all $|\eta| \in [0, \eta_0]$, $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}^+_0$, and $t \geq t_0$, the limit (5.29) equals

$$\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}(t) = \sum_{k\in\mathbb{N}} \sum_{x,z\in\mathfrak{L},|z|\leq 1} \langle\mathfrak{e}_x, (\Delta_{\mathrm{d}}+\lambda V_\omega)\mathfrak{e}_{x+z}\rangle i^k \int_{t_0}^{t} \mathrm{d}s_1 \cdots \int_{t_0}^{s_{k-1}} \mathrm{d}s_k \; \varrho^{(\beta,\omega,\lambda)}\left(\left[W^{\eta\mathbf{A}_l}_{s_k-t_0,s_k},\ldots,W^{\eta\mathbf{A}_l}_{s_1-t_0,s_1},\tau^{(\omega,\lambda)}_{t-t_0}(a^*_x a_{x+z})\right]^{(k+1)}\right). \tag{5.43}$$

This series is absolutely convergent, by Lemma 5.10. This proves Theorem 3.4(i) because of Corollary 5.7.

By Corollary 5.9, recall that, for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}^+_0$, $\mathbf{A} \in \mathbf{C}^\infty_0$, and $t \geq t_0$, $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}(t)$ is a real analytic function of $\eta \in \mathbb{R}$. Now, we use (5.43) to bound the Taylor coefficients of the function $\eta \mapsto \mathbf{S}^{(\omega,\eta\mathbf{A}_l)}(t)$ at $\eta = 0$; i.e., we prove Theorem 3.4(ii):

LEMMA 5.11 (Analytic Norm of the Internal Energy Increment). *For any* $\mathbf{A} \in \mathbf{C}^\infty_0$, *there exist* $\eta_1, D, \varepsilon \in \mathbb{R}^+$ *that depend on* $\mathbf{A}$ *such that, for all* $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}^+_0$, *and* $t \geq t_0$,

$$\sum_{m=0}^{\infty} \frac{\eta_1^m}{m!} \sup_{\eta\in[-\varepsilon,\varepsilon]} \left|\partial^m_\eta \mathbf{S}^{(\omega,\eta\mathbf{A}_l)}(t)\right| \leq D l^d .$$

PROOF. Similar to the derivation of (5.42), for any $\mathbf{A} \in \mathbf{C}_0^\infty$, there are constants $\eta_1, D, \varepsilon \in \mathbb{R}^+$ such that, for any $L, l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k \in \mathbb{N}$, $t \in (t_0, t_1]$, and $s_1, \ldots, s_k \in [t_0, t]$,

$$\sum_{x,z \in \mathfrak{L}, |z| \leq 1} \sum_{m=0}^{\infty} \frac{\eta_1^m}{m!} \sup_{\eta \in [-\varepsilon, \varepsilon]} \Big| \partial_\eta^m \big\{ \varrho^{(\beta,\omega,\lambda)} \big(\big[W_{s_k - t_0, s_k}^{\eta \mathbf{A}_l}, \ldots$$

$$\ldots, W_{s_1 - t_0, s_1}^{\eta \mathbf{A}_l}, \tau_{t-t_0}^{(\omega,\lambda)}(a_x^* a_{x+z}) \big]^{(k+1)} \big) \big\} \Big|$$

$$\leq \frac{D l^d k!}{2^{k-1} \big(t_1 - t_0 \big)^{k-1}}.$$

Now, use (5.43) together with fact that the η -derivative ∂_η is a closed operator with respect to to the norm of uniform convergence to arrive at the assertion. □

Appendix: Relative Entropy—Thermodynamic Limit

We give in the first subsection a concise account on the relative entropy in C^*-algebras. In the second subsection we show that the properties of the infinite fermion system result from features of the finite-volume one, at large volume.

A.1 Quantum Relative Entropy

Spacial Derivative Operator

Although the relative entropy can be defined for states on general C^*-algebras, it is natural to start with the special case of von Neumann algebras, which are (generally) noncommutative analogues of the algebra of bounded measurable functions. The definition of quantum relative entropy also requires the concept of the *spatial derivative* operator. The latter was first introduced by Connes [10] as a generalization of the relative modular operator. It is the noncommutative analogue of the Radon-Nikodym derivative of two measures defined as follows.

Let $\rho \in \mathfrak{M}^*$ be any normal state of a von Neumann algebra $\mathfrak{M}$ acting on a Hilbert space $\mathcal{H}$. We denote the so-called *lineal* of ρ by

$$\mathcal{D}_\rho := \big\{ \psi \in \mathcal{H} : \langle \psi, b b^* \psi \rangle_{\mathcal{H}} \leq D_\psi \rho\big(b b^*\big) \text{ for all } b \in \mathfrak{M} \text{ and some } D_\psi \in \mathbb{R}^+ \big\}. \tag{A.1}$$

Similar to [10, lemma 2], which is restricted to faithful states, this subspace of $\mathcal{H}$ is dense in supp(ρ). Here, by abuse of notation, supp(ρ) is defined to be either the smallest projection P such that $\rho(\mathrm{P}) = 1$ or the range of this projection P.

Let $(\mathcal{H}_\rho, \pi_\rho, \Psi_\rho)$ be the GNS representation of the state ρ. For any $\psi \in \mathcal{D}_\rho$, there is a bounded operator $R_\rho(\psi) : \mathcal{H}_\rho \to \mathcal{H}$ such that

$$R_\rho(\psi) \pi_\rho(b) \Psi_\rho = b\psi, \quad b \in \mathfrak{M}. \tag{A.2}$$

Clearly, for any $b \in \mathfrak{M}$, $bR_\rho(\psi) = R_\rho(\psi)\pi_\rho(b)$. This yields

$$\Theta_\rho(\psi, \tilde{\psi}) := R_\rho(\psi) R_\rho(\tilde{\psi})^* \in \mathfrak{M}', \quad \psi, \tilde{\psi} \in \mathcal{D}_\rho.$$

Let ϖ be a fixed normal state on $\mathfrak{M}'$. For any $\psi, \tilde{\psi} \in \mathcal{D}_\rho$ and $\psi_\perp, \tilde{\psi}_\perp \in \mathcal{D}_\rho^\perp$, we define the quadratic form q by

$$q_{\varpi,\rho}(\psi + \psi_\perp, \tilde{\psi} + \tilde{\psi}_\perp) := \varpi(\Theta_\rho(\psi, \tilde{\psi})). \tag{A.3}$$

Similar to what it is done in [10, lemmata 5 and 6], where the state ρ is faithful, $q_{\varpi,\rho}$ is a positive, densely defined quadratic form. Moreover, it is closable. In particular, by [24, theorem VIII.15], there is a unique positive self-adjoint operator $\partial_\rho \varpi$ acting on $\mathcal{H}$ such that the domain $\mathrm{Dom}(q)$ is a core for $(\partial_\rho \varpi)^{1/2}$ and

$$q_{\varpi,\rho}(\psi, \psi) = \langle (\partial_\rho \varpi)\psi, \psi \rangle_{\mathcal{H}} < \infty, \quad \psi \in \mathrm{Dom}(q).$$

Let $\mathrm{supp}(\partial_\rho \varpi)$ be the orthogonal projection on the range of $\partial_\rho \varpi$. From [22, eq. (4.4)],

$$\mathrm{supp}(\partial_\rho \varpi) = \mathrm{supp}(\varpi)\,\mathrm{supp}(\rho). \tag{A.4}$$

$\partial_\rho \varpi$ is named the *spatial derivative* operator and can be seen as a *noncommutative Radon-Nikodym derivative*; see [10]. For instance, at fixed state ρ, it is additive in ϖ. Since $\mathfrak{M}$ and $\mathfrak{M}'$ have symmetric roles, the spatial derivative $\partial_\varpi \rho$ can be defined as well and one finds that

$$\partial_\varpi \rho = (\partial_\rho \varpi)^{-1} \tag{A.5}$$

under the convention that, for any operator B, $B^{-1} \equiv 0$ on the subspace where $B = 0$. Moreover, as it is explained in [22, chap. 4], for faithful states, $\partial_\rho \varpi$ is nothing other than the *relative modular operator* $\mathbf{\Delta}(\varpi, \rho)$.

Relative Entropy for States on C^*-Algebras

Let $\mathcal{X}$ be a C^*-algebra and $\rho_2 \in \mathcal{X}^*$ be any reference state with GNS representation $(\mathcal{H}_{\rho_2}, \pi_{\rho_2}, \Psi_{\rho_2})$. Let $\tilde{\rho}_2 \in \mathfrak{M}^*$ be the normal state of the von Neumann algebra $\mathfrak{M} := \pi_{\rho_2}(\mathcal{X})''$ that is defined by extension from $\rho_2 \in \mathcal{X}^*$. Take any state $\rho_1 \in \mathcal{X}^*$ that is *quasi-contained* in ρ_2; that is, there exists a normal state $\tilde{\rho}_1 \in \mathfrak{M}^*$ such that

$$\tilde{\rho}_1(\pi_{\rho_2}(B)) = \rho_1(B), \quad B \in \mathcal{X}.$$

Then, by [6, theorems 2.4.21 and 2.5.31], there is $\Psi_{\rho_1} \in \mathcal{H}_{\rho_2}$ such that

$$\tilde{\rho}_1(\pi_{\rho_2}(B)) = \langle \Psi_{\rho_1}, \pi_{\rho_2}(B)\Psi_{\rho_1} \rangle_{\mathcal{H}_{\rho_2}}, \quad B \in \mathcal{X}. \tag{A.6}$$

Moreover, $\Psi_{\rho_1} \in \mathcal{H}_{\rho_2}$ induces a vector state $\tilde{\rho}_1'$ on the commutant $\mathfrak{M}'$ of $\mathfrak{M}$. Then, from (A.1) and (A.2), observe that $\mathcal{D}_{\tilde{\rho}_1'} = \mathfrak{M}\Psi_{\rho_1}$,

$$R_\rho(b\Psi_{\rho_1})\pi_\rho(b')\Psi_\rho = b(b'\Psi_{\rho_1}), \quad b' \in \mathfrak{M}',\ b \in \mathfrak{M}, \tag{A.7}$$

and the spatial derivative operator $\partial_{\widetilde{\rho}'_1}\widetilde{\rho}_2$ is a well-defined positive self-adjoint operator acting on $\mathcal{H}_{\rho_2}$. By (A.4), its support, seen as an orthogonal projection, equals

$$\mathrm{supp}(\partial_{\widetilde{\rho}'_1}\widetilde{\rho}_2) = \mathrm{supp}(\widetilde{\rho}'_1). \tag{A.8}$$

Then, Araki's definition of relative entropy takes the following form:

$$S_{\mathcal{X}}(\rho_1|\rho_2) := -\left\langle \ln(\partial_{\widetilde{\rho}'_1}\widetilde{\rho}_2)\Psi_{\rho_1}, \Psi_{\rho_1}\right\rangle_{\mathcal{H}_{\rho_2}} = -\rho_1(\ln(\partial_{\widetilde{\rho}'_1}\widetilde{\rho}_2)) \in \mathbb{R}_0^+; \tag{A.9}$$

see [22, eq. (5.1)]. This definition is sound because of (A.8) and

$$\Psi_{\rho_1} = \mathrm{supp}\big(\widetilde{\rho}'_1\big)\Psi_{\rho_1}.$$

If the state $\rho_1 \in \mathcal{U}^*$ is *not* quasi-contained in ρ_2, then the relative entropy of ρ_1 with respect to ρ_2 is by definition infinite, i.e., $S_{\mathcal{X}}(\rho_1|\rho_2) := +\infty$. However, this case never appears in this paper. By the Uhlmann monotonicity theorem [22, theorem 5.3], note that this definition does not depend on the choice of the vector $\Psi_{\rho_1} \in \mathcal{H}_{\rho_2}$ representing $\widetilde{\rho}_1$ via (A.6).

The *quantum* relative entropy $S_{\mathcal{X}}$ is the analogue of the relative entropy defined for probability measures on a Polish space. Compare formally (A.5) and (A.9) with [11, eq. (6.2.8)]. The positivity of the relative entropy as well as the equivalence relation between the two assertions $S_{\mathcal{X}}(\rho_1|\rho_2) = 0$ and $\rho_1 = \rho_2$ both follow from [22, theorem 5.5]. However, as for probability measures, neither $S_{\mathcal{X}}$ nor its symmetric version is a metric.

Relative Entropy for States on Full Matrix Algebras

In the case where $\mathcal{X}$ is a full matrix algebra $\mathcal{B}(\mathbb{C}^n)$ for some $n \in \mathbb{N}$, the relative entropy $S_{\mathcal{X}}$ has a simple explicit expression. Note that any finite-dimensional C^*-algebra is isomorphic to a direct sum of full matrix algebras and Lemma A.1 has a straighforward generalization to that case.

We denote by "tr" the normalized trace of $\mathcal{B}(\mathbb{C}^n)$. For any state $\rho \in \mathcal{B}(\mathbb{C}^n)^*$, there is a unique adjusted density matrix $\mathrm{d}_\rho \in \mathcal{B}(\mathbb{C}^n)$, that is, $\mathrm{d}_\rho \geq 0$, $\mathrm{tr}(\mathrm{d}_\rho) = 1$, and $\rho(A) = \mathrm{tr}(\mathrm{d}_\rho A)$ for all $A \in \mathcal{B}(\mathbb{C}^n)$; see [4, lemma 3.1(i)]. Then, by using an explicit GNS representation of ρ_2, one can explicitly compute the spatial derivative operator $\partial_{\widetilde{\rho}'_1}\widetilde{\rho}_2$ and, under the convention $x \ln x|_{x=0} := 0$, one explicitly finds the relative entropy $S_{\mathcal{B}(\mathbb{C}^n)}$ of any state $\rho_1 \in \mathcal{B}(\mathbb{C}^n)^*$ with respect to $\rho_2 \in \mathcal{B}(\mathbb{C}^n)^*$:

LEMMA A.1 (Relative Entropy—Finite-Dimensional Case). *Let $n \in \mathbb{N}$. For any state $\rho_1, \rho_2 \in \mathcal{B}(\mathbb{C}^n)^*$, the relative entropy $S_{\mathcal{B}(\mathbb{C}^n)}$ defined by* (A.9) *is equal to*

$$S_{\mathcal{B}(\mathbb{C}^n)}(\rho_1|\rho_2) = \begin{cases} \rho_1(\ln \mathrm{d}_{\rho_1} - \ln \mathrm{d}_{\rho_2}) \in \mathbb{R}_0^+ & \textit{if } \mathrm{supp}(\rho_2) \geq \mathrm{supp}(\rho_1), \\ +\infty & \textit{otherwise}. \end{cases}$$

PROOF. We give the proof for completeness and because it is instructive. Take two states $\rho_1, \rho_2 \in \mathcal{B}(\mathbb{C}^n)^*$. If ρ_1 is not quasi-contained in ρ_2, then clearly $\mathrm{supp}(\rho_2) \not\geq \mathrm{supp}(\rho_1)$ and $S_{\mathcal{B}(\mathbb{C}^n)}(\rho_1|\rho_2) = +\infty$.

Assume with loss of generality that ρ_2 is faithful. (Otherwise, one has to take a subspace of $\mathcal{B}(\mathbb{C}^n)$.) In particular, any state ρ_1 is quasi-contained in ρ_2. The GNS representation $(\mathcal{H}_{\rho_2}, \pi_{\rho_2}, \Psi_{\rho_2})$ of ρ_2 is, in this case, explicitly given as follows: $\mathcal{H}_{\rho_2}$ corresponds to the linear space $\mathcal{B}(\mathbb{C}^n)$ endowed with the Hilbert-Schmidt scalar product

$$\langle A, B\rangle_{\mathcal{H}_{\rho_2}} := \mathrm{Trace}_{\mathbb{C}^n}(A^* B), \quad A, B \in \mathcal{B}(\mathbb{C}^n). \tag{A.10}$$

It is convenient to define left and right multiplication operators on $\mathcal{B}(\mathbb{C}^n)$: For any $A \in \mathcal{B}(\mathbb{C}^n)$ we define the linear operators $\underset{\rightarrow}{A}$ and $\underset{\leftarrow}{A}$ acting on $\mathcal{B}(\mathbb{C}^n)$ by

$$B \mapsto \underset{\rightarrow}{A} B := AB \quad \text{and} \quad B \mapsto \underset{\leftarrow}{A} B := BA. \tag{A.11}$$

The representation π_{ρ_2} is the left multiplication, i.e.,

$$\pi_{\rho_2}(A) := \underset{\rightarrow}{A}, \quad A \in \mathcal{B}(\mathbb{C}^n).$$

The cyclic vector of the GNS representation of ρ_2 is defined by using the density matrix $\mathrm{D}_{\rho_2} \in \mathcal{B}(\mathbb{C}^n)$ of ρ_2 as

$$\Psi_{\rho_2} := \mathrm{D}_{\rho_2}^{1/2} \in \mathcal{H}_{\rho_2}. \tag{A.12}$$

The GNS representation $(\mathcal{H}_{\rho_2}, \pi_{\rho_2}, \Psi_{\rho_2})$ is known in the literature as the *standard representation* of the state ρ_2. See [12, sec. 5.4].

Let the "left" and "right" von Neumann algebras be respectively defined by

$$\underset{\rightarrow}{\mathfrak{M}} := \{\underset{\rightarrow}{A} \ : A \in \mathcal{B}(\mathbb{C}^n)\} = \pi_{\rho_2}(\mathcal{B}(\mathbb{C}^n))$$

and

$$\underset{\leftarrow}{\mathfrak{M}} := \{\underset{\leftarrow}{A} \ : A \in \mathcal{B}(\mathbb{C}^n)\} = \underset{\rightarrow}{\mathfrak{M}}'.$$

For any state $\rho_1 \in \mathcal{B}(\mathbb{C}^n)^*$, there is $\Psi_{\rho_1} := \mathrm{D}_{\rho_1}^{1/2} \in \mathcal{H}_{\rho_2}$ such that

$$\rho_1(B) = \left\langle \Psi_{\rho_1}, \pi_{\rho_2}(B)\Psi_{\rho_1}\right\rangle_{\mathcal{H}_{\rho_2}}, \quad B \in \mathcal{B}(\mathbb{C}^n). \tag{A.13}$$

In fact, $\mathrm{D}_{\rho_1} \in \mathcal{B}(\mathbb{C}^n)$ is the density matrix of ρ_1. Moreover, $\Psi_{\rho_1} \in \mathcal{H}_{\rho_2}$ induces a vector state ρ_1' on the commutant $\underset{\leftarrow}{\mathfrak{M}}$ of $\underset{\rightarrow}{\mathfrak{M}}$:

$$\rho_1'(\underset{\leftarrow}{A}) := \langle \Psi_{\rho_1}, \underset{\leftarrow}{A}\Psi_{\rho_1}\rangle_{\mathcal{H}_{\rho_2}} = \rho_1(A), \quad \underset{\leftarrow}{A} \in \underset{\leftarrow}{\mathfrak{M}}. \tag{A.14}$$

Its GNS representation is obviously given by $\mathcal{H}_{\rho_1'} := \mathrm{supp}(\rho_1)\mathcal{H}_{\rho_2}$ endowed with the scalar product (A.10), $\pi_{\rho_1'} := \mathbf{1}_{\underset{\leftarrow}{\mathfrak{M}}}$ and $\Psi_{\rho_1'} = \Psi_{\rho_1} = \mathrm{D}_{\rho_1}^{1/2}$. Moreover, $\mathcal{D}_{\rho_1'} = \underset{\rightarrow}{\mathfrak{M}}\Psi_{\rho_1}$ and, for any $\underset{\rightarrow}{A} \in \underset{\rightarrow}{\mathfrak{M}}$, the bounded operator $R_{\rho_1'}(\underset{\rightarrow}{A}\Psi_{\rho_1})$ defined by (A.2) equals in this case $\underset{\rightarrow}{A}$; see (A.7). Note that $\Psi_{\rho_2} \in \mathcal{H}_{\rho_2}$ induces a vector state ρ_2' on the commutant $\underset{\rightarrow}{\mathfrak{M}}$ of $\underset{\leftarrow}{\mathfrak{M}}$:

$$\rho_2'(\underset{\rightarrow}{A}) := \langle \Psi_{\rho_2}, \underset{\rightarrow}{A}\Psi_{\rho_2}\rangle_{\mathcal{H}_{\rho_2}} = \rho_2(A), \quad \underset{\rightarrow}{A} \in \underset{\rightarrow}{\mathfrak{M}}.$$

Then, using the cyclicity of the trace, we obtain that the quadratic form $q_{\widetilde{\rho}_2,\rho_1'}$ defined by (A.3) equals

$$q_{\widetilde{\rho}_2,\rho_1'}(\psi+\psi_\perp,\widetilde{\psi}+\widetilde{\psi}_\perp)=\langle\widetilde{\psi},\underleftarrow{\mathrm{D}_{\rho_1}^{-1}}\underrightarrow{\mathrm{D}_{\rho_2}}\psi\rangle_{\mathcal{H}_{\rho_2}}$$

for any $\psi,\widetilde{\psi}\in\mathcal{D}_{\rho_1'}$ and $\psi_\perp,\widetilde{\psi}_\perp\in\mathcal{D}_{\rho_1'}^\perp$. In particular, the spatial derivative $(\partial_{\rho_1'}\widetilde{\rho}_2)$ on the subspace $\operatorname{supp}(\rho_1)=\underrightarrow{\mathfrak{M}}\Psi_{\rho_1}$ is equal to

$$\partial_{\rho_1'}\rho_2=\underleftarrow{\mathrm{D}_{\rho_1}^{-1}}\,\underrightarrow{\mathrm{D}_{\rho_2}}.$$

Since $\underleftarrow{\mathfrak{M}}=\underrightarrow{\mathfrak{M}}'$, we observe that, on the subspace $\operatorname{supp}(\rho_1)=\underrightarrow{\mathfrak{M}}\Psi_{\rho_1}$,

$$\ln(\partial_{\rho_1'}\rho_2)=\ln\underrightarrow{\mathrm{D}_{\rho_2}}-\ln\underleftarrow{\mathrm{D}_{\rho_1}}=\underrightarrow{\ln\mathrm{D}_{\rho_2}}-\underleftarrow{\ln\mathrm{D}_{\rho_1}}.$$

By combining this equality with (A.9), (A.13), and (A.14), we arrive at

$$S_{\mathcal{B}(\mathbb{C}^n)}(\rho_1|\rho_2)=\mathrm{Trace}_{\mathbb{C}^n}(\mathrm{D}_{\rho_1}(\ln\mathrm{D}_{\rho_1}-\ln\mathrm{D}_{\rho_2}))\in\mathbb{R}_0^+. \qquad \square$$

A.2 Infinite System as Thermodynamic Limit

We present here the infinite system considered above as the thermodynamic limit of finite-volume systems. The aim is to show that all properties of the infinite model result from the corresponding ones of the finite-volume system at large volume.

Finite-Volume Free Fermion Systems on the Lattice

First, fix $L\in\mathbb{R}^+$ and recall that Λ_L is the box (3.2) of side length $2[L]+1$. Let

$$\big[\Delta_{\mathrm{d}}^{(L)}(\psi)\big](x):=2d\psi(x)-\sum_{\substack{|z|=1\\x+z\in\Lambda_L}}\psi(x+z),\quad x\in\Lambda_L,\ \psi\in\ell^2(\Lambda_L),$$

be, up to a minus sign, the discrete Laplacian restricted to the box Λ_L. For any $\omega\in\Omega$, we denote by $V_\omega^{(L)}$ the restriction of V_ω to $\ell^2(\Lambda_L)\subset\ell^2(\mathfrak{L})$:

$$V_\omega^{(L)}(\mathfrak{e}_x):=\mathbf{1}[x\in\Lambda_L]V_\omega(\mathfrak{e}_x),\quad x\in\mathfrak{L}.$$

Recall that $\{\mathfrak{e}_x\}_{x\in\mathfrak{L}}$ is the canonical orthonormal basis $\mathfrak{e}_x(y)\equiv\delta_{x,y}$ of $\ell^2(\mathfrak{L})$. Then, for any $\omega\in\Omega$ and $\lambda\in\mathbb{R}_0^+$, define the bounded self-adjoint operator

$$h_L^{(\omega,\lambda)}:=\Delta_{\mathrm{d}}^{(L)}+\lambda V_\omega^{(L)}\in\mathcal{B}(\ell^2(\Lambda_L)). \tag{A.15}$$

Obviously, this operator can also be extended to a bounded operator $\widetilde{h}_L^{(\omega,\lambda)}$ on $\ell^2(\mathfrak{L})$ by defining

$$\widetilde{h}_L^{(\omega,\lambda)}(\mathfrak{e}_x):=\begin{cases}h_L^{(\omega,\lambda)}(\mathfrak{e}_x) & \text{for } x\in\Lambda_L,\\ 0 & \text{for } x\in\mathfrak{L}\setminus\Lambda_L.\end{cases}$$

Since $\mathcal{U}_{\Lambda_L}$ is isomorphic to the algebra of all bounded linear operators on the fermion Fock space

$$\mathcal{F}:=\bigwedge(\ell^2(\Lambda_L)),$$

the Hamiltonian (3.5), that is,

$$H_L^{(\omega,\lambda)} = \sum_{x,y\in\Lambda_L} \langle \mathfrak{e}_x, h_L^{(\omega,\lambda)} \mathfrak{e}_y \rangle a_x^* a_y \in \mathcal{U}_{\Lambda_L}, \tag{A.16}$$

can be seen as the *second quantization* of $h_L^{(\omega,\lambda)}$ for all $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$. It is well-known in this case that the one-parameter (Bogoliubov) group $\tau^{(\omega,\lambda,L)} := \{\tau_t^{(\omega,\lambda,L)}\}_{t\in\mathbb{R}}$ of automorphisms uniquely defined by the condition

$$\tau_t^{(\omega,\lambda,L)}(a(\psi)) = a(\mathrm{e}^{it\widetilde{h}_L^{(\omega,\lambda)}}(\psi)), \quad t \in \mathbb{R}, \ \psi \in \ell^2(\mathfrak{L}),$$

(cf. [7, theorem 5.2.5]) satisfies

$$\tau_t^{(\omega,\lambda,L)}(B) = \mathrm{e}^{itH_L^{(\omega,\lambda)}} B \mathrm{e}^{-itH_L^{(\omega,\lambda)}}, \quad B \in \mathcal{U},$$

for each $L \in \mathbb{R}^+$ and all $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$.

Let $\varrho^{(\beta,\omega,\lambda,L)}$ be the unique $(\tau^{(\omega,\lambda,L)}, \beta)$-KMS state for any $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$ at fixed inverse temperature $\beta \in \mathbb{R}^+$. It is again well-known that this state is directly related with the Gibbs state $\mathfrak{g}^{(\beta,\omega,\lambda,L)}$ associated with the Hamiltonian $H_L^{(\omega,\lambda)}$ and defined by

$$\mathfrak{g}^{(\beta,\omega,\lambda,L)}(B) := \mathrm{Trace}_{\mathcal{F}}\left(B \frac{\mathrm{e}^{-\beta H_L^{(\omega,\lambda)}}}{\mathrm{Trace}_{\mathcal{F}}\left(\mathrm{e}^{-\beta H_L^{(\omega,\lambda)}}\right)} \right), \quad B \in \mathcal{U}_{\Lambda_L}, \tag{A.17}$$

for any $L, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, and $\lambda \in \mathbb{R}_0^+$. Indeed,

$$\varrho^{(\beta,\omega,\lambda,L)}(B_1 B_2) = \mathfrak{g}^{(\beta,\omega,\lambda,L)}(B_1)\,\mathrm{tr}(B_2), \quad B_1 \in \mathcal{U}_{\Lambda_L}, \ B_2 \in \mathcal{U}_{\mathfrak{L}\setminus\Lambda_L}, \tag{A.18}$$

where tr is the normalized trace (state) on $\mathcal{U}$. Note that tr is also named *tracial state* and satisfies a product property; see [4, sec. 4.2]. Here, $\mathcal{U}_{\mathfrak{L}\setminus\Lambda_L} \subset \mathcal{U}$ is the C^*-algebra generated by $\{a_x\}_{x\in\mathfrak{L}\setminus\Lambda_L}$ and the identity. In particular,

$$\varrho^{(\beta,\omega,\lambda,L)}(B) = \mathfrak{g}^{(\beta,\omega,\lambda,L)}(B), \quad B \in \mathcal{U}_{\Lambda_L}.$$

Let $\mathbf{A} \in \mathbf{C}_0^\infty$. For any sufficiently large $L \in \mathbb{R}^+$, $W_t^{\mathbf{A}} \in \mathcal{U}_{\Lambda_L}$. Therefore, consider the following finite-dimensional initial value problem on the space $\mathcal{B}(\mathcal{U}_{\Lambda_L})$ of bounded operators on $\mathcal{U}_{\Lambda_L}$ for any sufficiently large $L \in \mathbb{R}^+$:

(A.19) $\forall s, t \in \mathbb{R}, \ t \geq s$:

$$\partial_t \tau_{t,s}^{(\omega,\lambda,\mathbf{A},L)} = \tau_{t,s}^{(\omega,\lambda,\mathbf{A},L)} \circ \delta_t^{(\omega,\lambda,\mathbf{A},L)}, \quad \tau_{s,s}^{(\omega,\lambda,\mathbf{A},L)} := \mathbf{1},$$

with $\mathbf{1}$ being here the identity in $\mathcal{U}_{\Lambda_L}$. Here, the infinitesimal generator $\delta_t^{(\omega,\lambda,\mathbf{A},L)}$ of $\tau_{t,s}^{(\omega,\lambda,\mathbf{A},L)}$ equals

$$\delta_t^{(\omega,\lambda,\mathbf{A},L)}(\cdot) := i\left[H_L^{(\omega,\lambda)} + W_t^{\mathbf{A}}, \cdot\right] \tag{A.20}$$

and is of course a bounded operator acting on $\mathcal{U}_{\Lambda_L}$. Therefore, using the Dyson-Phillips series one shows, analogously to Section 5.2, the existence of a strongly continuous two-parameter (quasi-free) family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A},L)}\}_{t\geq s}$ of automorphisms of the finite-dimensional C^*-algebra $\mathcal{U}_{\Lambda_L}$ satisfying (A.19). See [7, sec. 5.4.2., prop. 5.4.26.] which, for the finite-volume dynamics, gives similar results to Theorems 5.1 and 5.3, and Proposition 5.4.

Heat Production and Internal Energy Increment

Similar to Definition 3.1, for any $L,\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\mathbf{A} \in \mathbf{C}_0^\infty$, the heat production $\mathbf{Q}^{(\omega,\mathbf{A},L)} \equiv \mathbf{Q}^{(\beta,\omega,\lambda,\mathbf{A},L)}$ in the finite-volume fermion system is defined, for any $t \geq t_0$, by

$$\mathbf{Q}^{(\omega,\mathbf{A},L)}(t) := \beta^{-1} S_{\mathcal{U}_\Lambda}\big(\mathfrak{g}^{(\beta,\omega,\lambda,L)} \circ \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A},L)} \big| \mathfrak{g}^{(\beta,\omega,\lambda,L)}\big) \in [0,\infty]. \tag{A.21}$$

Here, $S_{\mathcal{U}_\Lambda}$ is the quantum relative entropy defined by (3.1).

Like (3.8)–(3.9), the internal energy increment $\mathbf{S}^{(\omega,\mathbf{A},L)} \equiv \mathbf{S}^{(\beta,\omega,\lambda,\mathbf{A},L)}$ and the electromagnetic potential energy $\mathbf{P}^{(\omega,\mathbf{A},L)} \equiv \mathbf{P}^{(\beta,\omega,\lambda,\mathbf{A},L)}$ in the finite-volume fermion system are respectively defined by

$$\mathbf{S}^{(\omega,\mathbf{A},L)}(t) := \mathfrak{g}^{(\beta,\omega,\lambda,L)}\big(\tau_{t,t_0}^{(\omega,\lambda,\mathbf{A},L)}\big(H_L^{(\omega,\lambda)}\big)\big) - \mathfrak{g}^{(\beta,\omega,\lambda,L)}\big(H_L^{(\omega,\lambda)}\big),$$
$$\mathbf{P}^{(\omega,\mathbf{A},L)}(t) := \mathfrak{g}^{(\beta,\omega,\lambda,L)}\big(\tau_{t,t_0}^{(\omega,\lambda,\mathbf{A},L)}\big(W_t^{\mathbf{A}}\big)\big),$$

for any $L,\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \geq t_0$. Using [7, lemma 5.4.27] one also obtains that

$$\mathbf{S}^{(\omega,\mathbf{A},L)}(t) + \mathbf{P}^{(\omega,\mathbf{A},L)}(t) = \int_{t_0}^{t} \mathfrak{g}^{(\beta,\omega,\lambda,L)}\big(\tau_{s,t_0}^{(\omega,\lambda,\mathbf{A},L)}\big(\partial_s W_s^{\mathbf{A}}\big)\big)\mathrm{d}s \tag{A.22}$$

with

$$\mathfrak{g}^{(\beta,\omega,\lambda,L)}\big(\tau_{s,t_0}^{(\omega,\lambda,\mathbf{A},L)}\big(\partial_s W_s^{\mathbf{A}}\big)\big)$$

being, as in (3.11), the infinitesimal work of the electromagnetic field at time $t \in \mathbb{R}$ on the finite-volume fermion system.

Similarly to Theorem 3.2, the internal energy increment and the heat production also coincide at finite volume:

THEOREM A.2 (Heat Production as Internal Energy Increment). *For any $L,\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and all $t \geq t_0$,*

$$\mathbf{Q}^{(\omega,\mathbf{A},L)}(t) = \mathbf{S}^{(\omega,\mathbf{A},L)}(t) \in \mathbb{R}_0^+.$$

PROOF. The arguments follow those of [15]. Note first that

$$\mathfrak{g}^{(\beta,\omega,\lambda,L)} \circ \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A},L)} \in \mathcal{U}_{\Lambda_L}^* \tag{A.23}$$

is a state with adjusted density matrix. Its von Neumann entropy is equal, up to a minus sign, to

$$S_{\mathcal{U}_\Lambda}\big(\mathfrak{g}^{(\beta,\omega,\lambda,L)} \circ \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A},L)} \big| \operatorname{tr}\big) = S_{\mathcal{U}_\Lambda}(\mathfrak{g}^{(\beta,\omega,\lambda,L)} | \operatorname{tr}) \tag{A.24}$$

for all $t \geq t_0$ because $\tau_{t,t_0}^{(\omega,\lambda,\mathbf{A},L)}$ is an automorphism on $\mathcal{U}_{\Lambda_L}$. Recall that we denote by tr the normalized trace on $\mathcal{U}_\Lambda$ and, by finite dimensionality, the relative entropy equals (3.1); see also Lemma A.1. Using (3.1), (A.17), and (A.24), we directly derive the equality

$$\mathbf{S}^{(\omega,\mathbf{A},L)}(t) = \beta^{-1} S_{\mathcal{U}_\Lambda}\big(\mathfrak{g}^{(\beta,\omega,\lambda,L)} \circ \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A},L)} | \mathfrak{g}^{(\beta,\omega,\lambda,L)}\big) =: \mathbf{Q}^{(\omega,\mathbf{A},L)}(t). \quad \square$$

Therefore, similar to Theorem 3.4(i), it is straightforward to write the heat production in terms of multicommutators: For any $L, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, and $t \geq t_0$,

$$\mathbf{Q}^{(\omega,\mathbf{A},L)}(t) = \sum_{k \in \mathbb{N}} \int_{t_0}^{t} \mathrm{d}s_1 \cdots \int_{t_0}^{s_{k-1}} \mathrm{d}s_k \; \mathbf{u}_k^{(\omega,\mathbf{A},L)}(s_1, \ldots, s_k, t), \tag{A.25}$$

with the finite-volume heat energy coefficient $\mathbf{u}_k^{(\omega,\mathbf{A},L)} \equiv \mathbf{u}_k^{(\beta,\omega,\lambda,\mathbf{A},L)}$ defined by

$$\begin{aligned} &\mathbf{u}_k^{(\omega,\mathbf{A},L)}(s_1, \ldots, s_k, t) \\ &\quad := \sum_{\substack{x,y \in \Lambda_L \\ |x-y| \leq 1}} i^k \langle \mathfrak{e}_x, h_L^{(\omega,\lambda)} \mathfrak{e}_y \rangle \\ &\qquad \times \mathfrak{g}^{(\beta,\omega,\lambda,L)}\big(\big[W_{s_k - t_0, s_k}^{(\mathbf{A},L)}, \ldots, W_{s_1 - t_0, s_1}^{(\mathbf{A},L)}, \tau_{t-t_0}^{(\omega,\lambda,L)}(a_x^* a_y)\big]^{(k+1)}\big) \end{aligned} \tag{A.26}$$

for any $k \in \mathbb{N}$, $L, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, $t \geq t_0$, and $s_1, \ldots, s_k \in [t_0, t]$. Similar to the definition (5.13) of $W_{t,s}^{\mathbf{A}}$, note that we use above the notation

$$W_{t,s}^{(\mathbf{A},L)} \equiv W_{t,s}^{(\omega,\lambda,\mathbf{A},L)} := \tau_t^{(\omega,\lambda,L)}(W_s^{\mathbf{A}}) \in \mathcal{U}$$

for any $t, s \in \mathbb{R}$ and $\mathbf{A} \in \mathbf{C}_0^\infty$. Theorem 3.4(ii) also holds at finite volume, uniformly with respect to $L \in \mathbb{R}^+$.

Thermodynamic Limit of the Finite-Volume System

We first summarize well-known results on the infinite-volume dynamics and thermal state:

THEOREM A.3 (Infinite-Volume Dynamics and Thermal State). *Let $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, and $\lambda \in \mathbb{R}_0^+$. Then*:

(i) *For any $t \in \mathbb{R}$, the localized (quasi-free) automorphism $\tau_t^{(\omega,\lambda,L)}$ strongly converges to $\tau_t^{(\omega,\lambda)}$ as $L \to \infty$.*

(ii) *The $(\tau^{(\omega,\lambda,L)}, \beta)$-KMS state $\varrho^{(\beta,\omega,\lambda,L)}$ converges to the $(\tau^{(\omega,\lambda)}, \beta)$-KMS state $\varrho^{(\beta,\omega,\lambda)}$ in the weak-$*$ topology as $L \to \infty$.*

PROOF. See [7, chaps. 5.2 and 5.3]. $\square$

Then, from Equation (A.22), Theorem A.3, and Lebesgue's dominated convergence theorem, it is clear that the energy increments $\mathbf{S}^{(\omega,\mathbf{A})}$ and $\mathbf{P}^{(\omega,\mathbf{A})}$, respectively defined by (3.8) and (3.9), result from the finite-volume energy increments $\mathbf{S}^{(\omega,\mathbf{A},L)}$ and $\mathbf{P}^{(\omega,\mathbf{A},L)}$:

COROLLARY A.4 (Energy Increments as Thermodynamic Limits). *For any* $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, *and all* $t \geq t_0$,

$$\mathbf{S}^{(\omega,\mathbf{A})}(t) = \lim_{L\to\infty} \mathbf{S}^{(\omega,\mathbf{A},L)}(t) \quad \textit{and} \quad \mathbf{P}^{(\omega,\mathbf{A})}(t) = \lim_{L\to\infty} \mathbf{P}^{(\omega,\mathbf{A},L)}(t).$$

By combining this with Theorems 3.2 and A.2 we show the same property for the heat production:

COROLLARY A.5 (Heat Production as Thermodynamic Limit). *For any* $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, *and all* $t \geq t_0$,

$$\mathbf{Q}^{(\omega,\mathbf{A})}(t) = \lim_{L\to\infty} \mathbf{Q}^{(\omega,\mathbf{A},L)}(t).$$

By Theorem 3.4, recall that, for any $\mathbf{A} \in \mathbf{C}_0^\infty$, there is a constant $\eta_0 \in \mathbb{R}^+$ such that, for all $|\eta| \in [0, \eta_0]$, $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $t \geq t_0$,

$$\mathbf{Q}^{(\omega,\eta\mathbf{A})}(t) = \sum_{k\in\mathbb{N}} \int_{t_0}^{t} \mathrm{d}s_1 \cdots \int_{t_0}^{s_{k-1}} \mathrm{d}s_k \, \mathbf{u}_k^{(\omega,\eta\mathbf{A})}(s_1, \ldots, s_k, t). \tag{A.27}$$

Here, the heat energy coefficient $\mathbf{u}_k^{(\omega,\mathbf{A})} \equiv \mathbf{u}_k^{(\beta,\omega,\lambda,\mathbf{A})}$ is defined, for any $k \in \mathbb{N}$, $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, $t \geq t_0$, and $s_1, \ldots, s_k \in [t_0, t]$, by

$$\begin{aligned}\mathbf{u}_k^{(\omega,\mathbf{A})}&(s_1, \ldots, s_k, t)\\ &:= \sum_{\substack{x,y\in\mathfrak{L}\\ |x-y|\leq 1}} i^k \langle \mathfrak{e}_x, (\Delta_{\mathrm{d}} + \lambda V_\omega)\mathfrak{e}_y\rangle \\ &\qquad \times \varrho^{(\beta,\omega,\lambda)}\big(\big[W^{\mathbf{A}}_{s_k-t_0,s_k}, \ldots, W^{\mathbf{A}}_{s_1-t_0,s_1}, \tau^{(\omega,\lambda)}_{t-t_0}(a_x^* a_y)\big]^{(k+1)}\big)\end{aligned}$$

with $W^{\mathbf{A}}_{t,s} := \tau_t^{(\omega,\lambda)}(W^{\mathbf{A}}_s) \in \mathcal{U}$ for any $t, s \in \mathbb{R}$; see (5.13). The series (A.27) absolutely converges for the above range of parameters.

Then, by combining these last series with (A.25)–(A.26), Theorem A.3 and Corollary A.5, one directly obtains the following result:

THEOREM A.6 (Taylor Coefficients of $\mathbf{Q}^{(\omega,\eta\mathbf{A})}$ as Thermodynamic Limit). *For any* $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, $t \geq t_0$, *and* $m \in \mathbb{N}$,

$$\begin{aligned}\partial_\eta^m \mathbf{Q}^{(\omega,\eta\mathbf{A})}(t)|_{\eta=0} &= \lim_{L\to\infty} \partial_\eta^m \mathbf{Q}^{(\omega,\eta\mathbf{A},L)}(t)|_{\eta=0}\\ &= \sum_{k\in\mathbb{N}} \int_{t_0}^{t} \mathrm{d}s_1 \cdots \int_{t_0}^{s_{k-1}} \mathrm{d}s_k \\ &\qquad \lim_{L\to\infty} \big\{\partial_\eta^m \mathbf{u}_k^{(\omega,\eta\mathbf{A},L)}(s_1, \ldots, s_k, t)\big|_{\eta=0}\big\},\end{aligned}$$

where the above series is absolutely convergent.

PROOF. The proof uses similar arguments to those showing Lemma 5.11. We omit the details. □

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J.-B. BRU
Departamento de Matemáticas
Facultad de Ciencia y Tecnología
Universidad del País Vasco
Apartado 644
48080 Bilbao
SPAIN
and
Ikerbasque
Basque Foundation for Science
48011 Bilbao
SPAIN
and
BCAM—Basque Center
for Applied Mathematics
Mazarredo, 14
48009 Bilbao
SPAIN
E-mail: jb.bru@ikerbasque.org

C. HERTLING
Institut für Mathematik
Universität Mainz
Staudingerweg 9
55099 Mainz
GERMANY
E-mail: carolin@familiehertling.de

W. DE SIQUEIRA PEDRA
Departamento de Física-Matemáticas
Instituto de Física
Universidade de São Paulo
05315-970 São Paulo/SP
BRAZIL
E-mail: wpedra@if.usp.br

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